Topological properties of XY model and its dual fermion model

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Abstract

We use the twist boundary condition method to study the topological properties of one dimensional spin-1/2 XY model and its dual model – one dimensional superconductors. We show that there is a topological degeneracy for one dimensional superconductors, and relate it to the degeneracy protected by Z_2 rotation symmetry of XY model. We also find a Z_2 topological invariant which can describe the phase transition in XY model.

1. Model

One dimensional spin-1/2 XY model:

$$H_{XY} = -\sum(\frac{1+\Delta}{2}\sigma_j^x\sigma_{j+1}^x + \frac{1-\Delta}{2}\sigma_j^y\sigma_{j+1}^y) - \lambda\sum\sigma_j^z \quad (1)$$

Where $\sigma_{x,y,z}$ are Pauli matrices.

- A standard quantum phase transition system which can be described by symmetry breaking.
- $|\lambda| < 1$, there is a two fold groundstate degeneracy protected by Z_2 rotational symmetry $U = \prod \exp(i\pi\sigma_j^z/2)$. When $|\lambda| > 1$, there is no groundstate degeneracy.

What's interesting is that, XY model can be mapped to a one dimensional superconductor via a Jordan-Wigner transformation:

$$\sigma_i^+ = 2\prod_{j=0}^{i-1}\sigma_j^z c_i^\dagger \qquad \sigma_i^- = 2\prod_{j=0}^{i-1}\sigma_j^z c_i \qquad \sigma_i^z = 2c_i^\dagger c_i - 1 \qquad \sigma_i^+ = \sigma_i^x + i\sigma_i^y \qquad \sigma_i^- = \sigma_i^x - i\sigma_i^y \quad (2)$$

After applying the Jordan-Wigner transformation, one can obtain:

$$H_{XY} = \sum_{j=1}^{N}\left[c_{j+1}^\dagger c_j + h.c. + \Delta(c_{j+1}^\dagger c_j^\dagger + h.c.) - \lambda(2c_j^\dagger c_j - 1)\right] - (1+T)(c_N^\dagger c_1 + c_1^\dagger c_N + \Delta c_1 c_N + \Delta c_N^\dagger c_1^\dagger) \quad (3)$$

Where $T = \prod_{i=1}^N \sigma_i^z$. It is easy to find that T commutes with the Hamiltonian Eq.1, and $T^2 = 1 \Longrightarrow T = \pm 1$. When $T = -1$, Eq.3 will be the one dimensional topological superconductor:

$$\begin{aligned} H &= \sum\left[c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} + \Delta(c_{j+1}^\dagger c_j^\dagger + c_j c_{j+1})\right] - 2\lambda\sum c_j^\dagger c_j \\ &= 2\sum_{k>0}(c_k^\dagger\, c_{-k})\left[(\cos k - \lambda)\sigma_z + \Delta\sin k\sigma_y\right](c_k, c_{-k}^\dagger)^T \end{aligned} \quad (4)$$

And then a topological invariant–winding number can be defined as below:

$$n = \frac{1}{2\pi i}\int q(k)dq^\dagger(k) \qquad q(k) = \frac{\cos ka - \lambda + i\Delta\sin ka}{\sqrt{(\cos ka - \lambda)^2 + \Delta^2\sin^2 ka}} \quad (5)$$

Calculation shows that when $|\lambda| < 1$, $n = 1$, while if $|\lambda| > 1$, $n = 0$.
This topological invariant is defined with the Bloch Hamiltonian, but what if there is **no translational symmetry**?

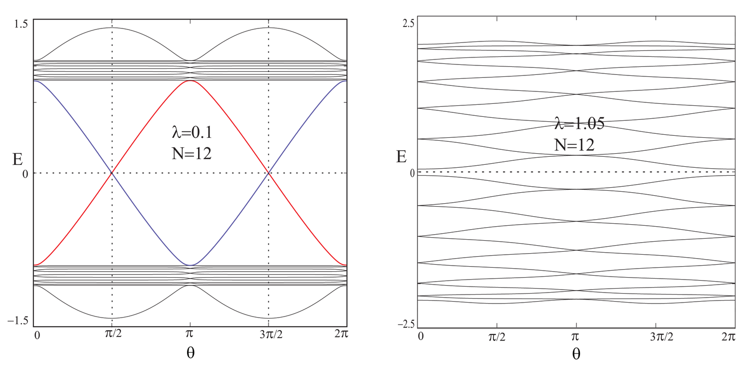
2. Twist Boundary Condition

2.1 1-dimensional Topological Superconductors

Using the twist boundary condition $c_{N+1} = e^{i\theta}c_1$, the Hamiltonian of one dimensional topological superconductor will be:

$$H = \sum_{j=1}^{N-1}(c_{j+1}^\dagger c_j + h.c.) + e^{-i\theta}c_1^\dagger c_N + h.c. - 2\lambda\sum_j c_j^\dagger c_j + \Delta\left[(\sum_{j=1}^{N-1}c_{j+1}^\dagger c_j^\dagger + c_j c_{j+1}) + e^{-i\theta}c_1^\dagger c_N^\dagger + e^{i\theta}c_N c_1\right] \quad (6)$$

Difficulty: Due to the pairing term $c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger$, the usual way, which define the topological invariant with twist parameter θ directly, doesn't work.



(a) (b)

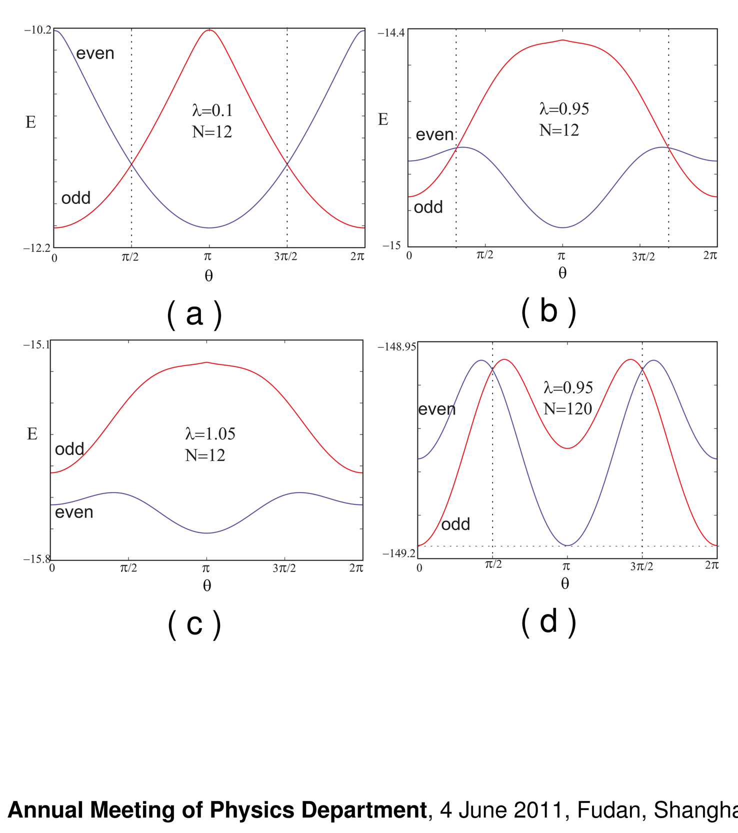
The energy spectrum for the quasi-particle of the twist BdG Hamiltonian(Eq.6). Due to particle-hole symmetry, the quasi-particle energy spectrum always comes in pairs. One can find that when $|\lambda| < 1$, there is always a level crossing between the two quasi-particle state nearest to 0(Fig.(a)). While when $|\lambda| > 1$, this level crossing disappears. This level crossing is topological, it always exists even under small disorder or interaction.

For periodic boundary condition(PBC) and anti-PBC($\theta = \pi$), the twist BdG Hamiltonian can be solved analytically. The BCS groundstate will be:

$$\begin{aligned} \psi_+ &= |0\rangle_0\prod_{k=1}^{M-1}\left[\cos\frac{\theta_k}{2}|0\rangle_k|0\rangle_{-k} - i\sin\frac{\theta_k}{2}|1\rangle_k|1\rangle_{-k}\right]|1\rangle_M & \theta = 0, \quad |\lambda| < 1 \\ \psi_+ &= |1\rangle_0\prod_{k=1}^{M-1}\left[\cos\frac{\theta_k}{2}|0\rangle_k|0\rangle_{-k} - i\sin\frac{\theta_k}{2}|1\rangle_k|1\rangle_{-k}\right]|1\rangle_M & \theta = 0, \quad \lambda > 1 \\ \psi_+ &= |0\rangle_0\prod_{k=1}^{M-1}\left[\cos\frac{\theta_k}{2}|0\rangle_k|0\rangle_{-k} - i\sin\frac{\theta_k}{2}|1\rangle_k|1\rangle_{-k}\right]|0\rangle_M & \theta = 0, \quad \lambda < -1 \\ \psi_- &= \prod_{k=0}^{M-1}(\cos\frac{\theta'_k}{2}|0\rangle'_k|0\rangle'_{-1-k} - i\sin\frac{\theta'_k}{2}|1\rangle'_k|1\rangle'_{-1-k}) & \theta = \pi, \quad \text{for any } \lambda \end{aligned} \quad (7)$$

Where $|1\rangle_k(|1\rangle'_k)$ stands for the state with occupation number of $c_k(c'_k)$ to be one, and

$$\begin{aligned} c_k &= \frac{1}{\sqrt{N}}\sum_{j=1}^{N}e^{-i\frac{2\pi k}{N}\cdot j}c_j & c_k &= \frac{1}{\sqrt{N}}\sum_{j=1}^{N}e^{-i(\frac{2\pi k}{N}+\frac{\pi}{N})\cdot j}c_j & 2M &= N \\ \cos\theta_k &= \frac{\lambda - \cos\frac{2\pi}{N}k}{\Lambda_k} & \sin\theta_k &= \frac{\Delta\sin\frac{2\pi}{N}k}{\Lambda_k} & \Lambda_k &= \sqrt{[\lambda - \cos\frac{2\pi}{N}k]^2 + \Delta^2\sin^2\frac{2\pi}{N}k} \\ \cos\theta'_k &= \frac{\lambda - \cos\frac{2\pi}{N}(k+1/2)}{\Lambda'_k} & \sin\theta'_k &= \frac{\Delta\sin\frac{2\pi}{N}(k+1/2)}{\Lambda'_k} & \Lambda'_k &= \sqrt{[\lambda - \cos\frac{2\pi}{N}(k+1/2)]^2 + \Delta^2\sin^2\frac{2\pi}{N}(k+1/2)} \end{aligned} \quad (8)$$



(c) (d)

From the groundstate solution and figure left, one can find that for anti-PBC($\theta = \pi$), the groundstate's fermion parity is even for any λ. While for PBC($\theta = \pi$), the groundstate's fermion parity is even when $|\lambda| > 1$, is odd when $|\lambda| < 1$. Thus when $|\lambda| < 1$, the groundstates of PBC and anti-PBC are totally different, and in the thermodynamic limit, the two groundstates will be a two fold degeneracy groundstates. This degeneracy is a topological degeneracy which will be robust under disorder and perturbation. While when $|\lambda| > 1$, there is no degeneracy for the groundstate.
Also one can observe a topological level crossing between the two lowest eigenstate when $|\lambda| < 1$. It should be noted that in the thermodynamic limit, the level crossing is at $\theta = \pi/2, 3\pi/2$. But when $|\lambda|$ is near to the critical point 1 and the system is small, the level crossing point differs from $\theta = \pi/2, 3\pi/2$.

2.2 One dimensional XY Model

With Jordan-Wigner transformation, the XY model can be transformed to Eq.3, which differs the one dimensional superconductor by a term $-(1+T)(c_N^\dagger c_1 + c_1^\dagger c_N + \Delta c_1 c_N + \Delta c_N^\dagger c_1^\dagger)$. When $T = -1$, Eq.3 is just the one dimensional superconductor with PBC. While if $T = 1$, Eq.3 will change to the one dimensional superconductor with anti-PBC. Meanwhile, $T = \pm 1$ just corresponds to the two fold degeneracy of XY model when $|\lambda| < 1$.

Therefore, the two-fold degeneracy of groundstate protected by Z_2 rotation symmetry of XY model just corresponds to the topological degeneracy of one dimensional superconductor.

Can we define a topological invariant for the XY model?

We impose the twist boundary condition on XY model: $\sigma_{N+1}^x = \cos\theta\sigma_1^x + \sin\theta\sigma_1^y$ and $\sigma_{N+1}^y = \cos\theta\sigma_1^y - \sin\theta\sigma_1^x$. This way of twist XY model will gives the same Hamiltonian of one dimensional twist superconductors(Eq.6). One obtains the Hamiltonian:

$$H(\theta) = -\sum_{j=1}^{N-1}(\frac{1+\Delta}{2}\sigma_j^x\sigma_{j+1}^x + \frac{1-\Delta}{2}\sigma_j^y\sigma_{j+1}^y) - \lambda\sum\sigma_j^z - \left[\frac{1+\Delta}{2}\sigma_N^x(\cos\theta\sigma_1^x + \sin\theta\sigma_1^y) + \frac{1-\Delta}{2}\sigma_N^y(\cos\theta\sigma_1^y - \sin\theta\sigma_1^x)\right] \quad (9)$$

And we apply a rotation $U(\phi) = \prod\exp(i\phi\sigma_j^z/4)$ to the XY model, the Hamiltonian will be $H(\phi,\theta) = U(\phi)^\dagger H(\theta)U(\phi)$. With two parameter θ, ϕ, one can define a topological invariant–Chern number. However, this Hamiltonian obeys a pseudo time-reversal symmetry: $KH(\theta,\phi)K = H(-\theta,-\phi)$, where K is just the complex conjugation.

Time-reversal symmetry$\Longrightarrow$Chern-number$= 0$

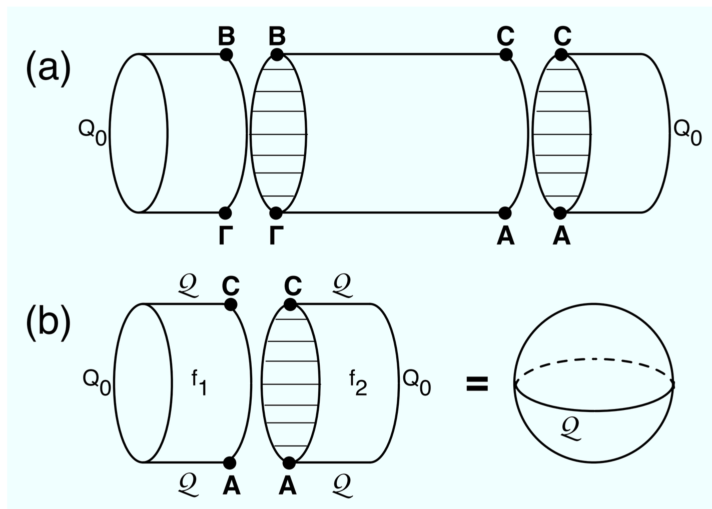
Z_2 topological invariant

Define a topological invariant with the half parameter space: $\theta \in [0,\pi]$, $\phi \in [0,2\pi)$. Mathematics theory tells us there should be a Z_2 topological invariant. But how to define it?

Obstacle to define such Z_2 topological invariant:

- Chern-number can only defined for a closed surface. For our system, there are boundary at $\theta = 0, \pi$, thus a Chern-number can't be defined directly.
- The time-reversal symmetry of the system satisfies $K^2 = 1$, not $T^2 = -1$.

We use a **"contraction"** method to define the Z_2 topological invariant.



Contract the two boundaries at $\theta = 0, \pi$ to a point, and then the two-dimensional surface will be a closed one. A Chern-number can be defined then. To guarantee this Chern-number is a Z_2 topological invariant, there should be a constraint on the "contraction".

Since the XY model has a two fold degeneracy when $|\lambda| < 1$, we use the two groundstate to define the topological invariant. One can obtain the exact wave function of the two state at $\theta = 0, \pi$:

$$\begin{aligned} \psi_1(\phi) &= \psi_{T=-1}(\phi,\theta=0) = |0\rangle_0\prod_{k=1}^{M-1}\left[\cos\frac{\theta_k}{2}|0\rangle_k|0\rangle_{-k} - ie^{i\phi}\sin\frac{\theta_k}{2}|1\rangle_k|1\rangle_{-k}\right]|1\rangle_M \\ \psi_2(\phi) &= \psi_{T=-1}(\phi,\theta=\pi) = |0\rangle'_0|1\rangle'_{-1}\prod_{k=1}^{M-1}(\cos\frac{\theta'_k}{2}|0\rangle'_k|0\rangle'_{-1-k} - ie^{i\phi}\sin\frac{\theta'_k}{2}|1\rangle'_k|1\rangle'_{-1-k}) \\ \psi_3(\phi) &= \psi_{T=1}(\phi,\theta=0) = \prod_{k=0}^{M-1}(\cos\frac{\theta'_k}{2}|0\rangle'_k|0\rangle'_{-1-k} - ie^{i\phi}\sin\frac{\theta'_k}{2}|1\rangle'_k|1\rangle'_{-1-k}) \\ \psi_4(\phi) &= \psi_{T=1}(\phi,\theta=\pi) = |1\rangle_0\prod_{k=1}^{M-1}\left[\cos\frac{\theta_k}{2}|0\rangle_k|0\rangle_{-k} - ie^{i\phi}\sin\frac{\theta_k}{2}|1\rangle_k|1\rangle_{-k}\right]|1\rangle_M \end{aligned} \quad (10)$$

Note that $\psi_2(\phi) = c'^\dagger_{-1}\psi_3(\phi)/\cos\frac{\theta'_0}{2}$, $\psi_4(\phi) = c_0^\dagger\psi_1(\phi)$. Thus we suppose the contraction $f(\eta), \eta \in [0,1]$ satisfies that:

$$\psi_i(f(0),\phi) = \psi_i(\phi) \qquad \psi_i(f(1),\phi) = Q_i \qquad \psi_2(f(\eta),\phi) = c'^\dagger_{-1}\psi_3(f(\eta),\phi)/\cos\frac{\theta'_0(f(\eta))}{2} \qquad \psi_4(f(\eta),\phi) = c_0^\dagger\psi_1(f(\eta),\phi) \quad (11)$$

Consider two different contraction $f_1(\eta), f_2(\eta)$ satisfies the above condition, then the Chern-number difference between two contractions will be $\delta n = \sum n_i$, where:

$$n_i = \frac{1}{2\pi i}\int(\langle\partial_\eta\psi_i|\partial_\phi\psi_i\rangle - \langle\partial_\phi\psi_i|\partial_\eta\psi_i\rangle)d\eta d\phi \qquad \psi_i(\phi,\eta) = \begin{cases}\psi_i(\phi, f_1(1-2\eta)) & \text{if } 0 \le \lambda \le 1/2 \\ \psi_i(\phi, f_2(2\eta-1)) & \text{if } 1/2 \le \lambda \le 1\end{cases} \quad (12)$$

Since f_1, f_2 satisfies the above condition, $n_1 = n_4$, $n_2 = n_3$. Therefore, δn is always a even integer, and we have obtained a Z_2 invariant.
One right contraction is $\Delta \to 0$. If $\lambda > 0$, the Z_2 invariant will be

$$n = \left(1 - \frac{\lambda - \cos\frac{\pi}{N}}{|\lambda - \cos\frac{\pi}{N}|}\right)/2 \quad (13)$$

Therefore,

$$n = \begin{cases}1 & \lambda < \cos\frac{\pi}{N} \\ 0 & \lambda > \cos\frac{\pi}{N}\end{cases} \quad (14)$$

For $N \to \infty$, the critical point is $\lambda = 1$.

3. Conclusion

We have used the twist boundary condition to study the topological properties of XY model and its dual fermion model. We find topological level crossing in the one dimensional topological superconductors, and we also relates the topological degeneracy in topological superconductors to the symmetry degeneracy in XY model. We also propose a Z_2 invariant to describe the quantum phase transition in XY model. It should be noted that this Z_2 topological invariant is well defined without using the language of Jordan-Wigner fermion. It is actually a topological invariant for spin model. And this Z_2 topological invariant is beyond the classification of topological band theory which is successful in discussing topological insulators and superconductors.

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