Chiral Kinetic Theory in Curved Space-time

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Abstract

We use the Wigner function method to calculate the chiral effects in curved space-time. The calculation is accomplished in a perturbative way with the assumption that the gravitational field is covariant constant field and the system is nearly local equilibrium. We obtain the chiral anormaly results that the vector and axial vector currents are in direct proportion to the fluid vorticity.

Introduction

Wigner function method is a bridge between microscopic and macroscopic regimes. The conception of this method is simple. Firstly, write down a function which is linked to observable macroscopic quantities, then find the equation that the function is satisfied in quantum way. Wigner function is in the form of Fourier representation. And the equation which Wigner function is satisfied can be decomposed into two equations, Liouville equation and mass-shell condition[1], by separating the real and imaginary parts of it.

Chiral Effects

For massless particles, chirality is the same as helicity: the sign of the projection of the spin vector onto the momentum vector.
Chiral chemical potential:

$$\begin{aligned} \mu_R &= \mu + \mu_5, \\ \mu_L &= \mu - \mu_5. \end{aligned} \tag{1}$$

The chiral chemical potential $\mu_5 \neq 0$.
Chiral effects for massless fermions

- Chiral magnetic effect(CME)

$$\mathbf{J} = \sigma_5 \mathbf{B}, \qquad \sigma_5 = \frac{e^2}{2\pi^2}\mu_5. \tag{2}$$

  Where the current $\mathbf{J}$ is defined by $\mathbf{J}^\mu = e\langle\bar\psi\gamma^\mu\psi\rangle$.

- Chiral separation effect(CSE)

$$\mathbf{J}_5 = \sigma \mathbf{B}, \qquad \sigma = \frac{e^2}{2\pi^2}\mu. \tag{3}$$

  Where the axial current $\mathbf{J}_5$ is defined by $\mathbf{J}_5^\mu = e\langle\bar\psi\gamma^\mu\gamma_5\psi\rangle$.

Wigner Function And Curved Space-time

Wigner operator:

$$\hat{W}_{\alpha\beta}(x,p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\bar\psi_\beta(x) e^{1/2 y\cdot\overleftarrow{\partial}} e^{-1/2 y\cdot\partial}\psi_\alpha(x). \tag{4}$$

Where the derivative $\overleftarrow{\partial}_\mu(\partial_\mu)$ acting to the left(right).
Wigner function:

$$W(x,p) \equiv \langle : \hat{W}(x,p) : \rangle. \tag{5}$$

RHS means that Wigner operator respect to vacuum state.
For gauge field or curved spacetime, just replace the partial derivatives in Wigner operator by the covariant derivatives.
Curved spacetime case:

$$\partial_\mu \longrightarrow \nabla_\mu \equiv \partial_\mu + \Gamma_\mu, \tag{6}$$

$$\overleftarrow{\partial}_\mu \longrightarrow \overleftarrow{\nabla}_\mu \equiv \overleftarrow{\partial}_\mu - \Gamma_\mu, \tag{7}$$

where $\Gamma_\mu \equiv -\frac{i}{4}\omega_\mu^{ab}\sigma_{ab}$ with $\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b]$ and ω_μ^{ab} the vierbein connection.
Vierbein: $e^a = e^a_\mu \partial^\mu$.
Then the currents can be written as

$$\mathbf{J}^\mu(x) = q\, tr \int d^4p \gamma^\mu W(x,p), \tag{8}$$

$$\mathbf{J}_5^\mu(x) = q\, tr \int d^4p \gamma^\mu\gamma^5 W(x,p). \tag{9}$$

In the covariant constant external gravitational field case, the Wigner function satisfies the equation

$$\gamma^\mu\left(\frac{i}{2}\nabla_\mu + p_\mu + iR_{\mu\nu}\frac{\partial_p^\nu}{2}\right)W(x,p) = i\gamma^\mu\frac{\partial_p^\nu}{8}[R_{\mu\nu}, W(x,y)], \tag{10}$$

with $R_{\mu\nu} = R_{\mu\nu}^{ab}\sigma_{ab} = i[\nabla_\mu, \nabla_\nu]$ the curvature.

Wigner Function Decomposing

Winger function decomposed in terms of 16 independent generators of the Clifford algebra

$$\begin{aligned} W(x,y) = &\frac{1}{4}[\mathcal{F}(x,p) + i\gamma^5\mathcal{P}(x,p) + \gamma^\mu\mathcal{V}_\mu(x,p) \\ &+ \gamma^5\gamma^\mu\mathcal{A}_\mu(x,p) + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}(x,p)]. \end{aligned} \tag{11}$$

Substitute the decomposed Wigner function (11) into kinetic equation (10), we obtain the restriction relations

$$\begin{aligned} (\partial_x^a + \frac{1}{2}\omega_b^{[ba]})\mathcal{V}_a &= \epsilon_{abcd}\eta^{ag}R_{gf}^{cd}\partial_p^f\mathcal{A}^b, \\ (\partial_x^a + \frac{1}{2}\omega_b^{[ba]})\mathcal{A}_a &= \epsilon_{abcd}\eta^{ag}R_{gf}^{cd}\partial_p^f\mathcal{V}^b, \end{aligned} \tag{12}$$

$$\begin{aligned} p^a\mathcal{V}_a &= \frac{1}{4}R_{af}^{[ab]}\partial_p^f\mathcal{V}_b, \\ p^a\mathcal{A}_a &= \frac{1}{4}R_{af}^{[ab]}\partial_p^f\mathcal{A}_b, \end{aligned} \tag{13}$$

$$\begin{aligned} \epsilon^{abcd}(\partial_c\mathcal{V}_d + \frac{1}{2}\omega_c^{[mn]}\mathcal{V}_n\eta_{md}) &= 2(p^a\mathcal{A}^b - p^b\mathcal{A}^a) + R_{mf}^{nc}\partial_p^f\mathcal{A}^d\epsilon_{dcng}^{abmg} \\ &\quad + \frac{1}{2}(\eta^{mb}R_{mf}^{[ac]}\partial_p^f\mathcal{A}_c - \eta^{ma}R_{mf}^{[bc]}\partial_p^f\mathcal{A}_c), \\ \epsilon^{abcd}(\partial_c\mathcal{A}_d + \frac{1}{2}\omega_c^{[mn]}\mathcal{A}_n\eta_{md}) &= 2(p^a\mathcal{V}^b - p^b\mathcal{V}^a) + R_{mf}^{nc}\partial_p^f\mathcal{V}^d\epsilon_{dcng}^{abmg} \\ &\quad + \frac{1}{2}(\eta^{mb}R_{mf}^{[ac]}\partial_p^f\mathcal{V}_c - \eta^{ma}R_{mf}^{[bc]}\partial_p^f\mathcal{V}_c), \end{aligned} \tag{14}$$

where $A^{[ab]} = A^{ab} - A^{ba}$.

Nearly Equilibrium System And the Rotating Frame

We assume that space-time derivative ∂_x and the connection ω_a^{bc} are small variables of the same order and can be as parameters in the power expansion of $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$

$$\mathcal{V}_\mu = \mathcal{V}_\mu^0 + \mathcal{V}_\mu^1 + \cdots, \tag{15}$$

$$\mathcal{A}_\mu = \mathcal{A}_\mu^0 + \mathcal{A}_\mu^1 + \cdots. \tag{16}$$

We assume the zeroth order take the equilibrium form[2]

$$\mathcal{V}_\mu^0 = p_\mu\delta(p^2)V^0, \qquad \mathcal{A}_\mu^0 = p_\mu\delta(p^2)A^0, \tag{17}$$

$$\begin{aligned} V^0 &= \sum_{s=\pm1}\theta(su\cdot p)(f_{s,R} + f_{s,L}), \\ A^0 &= \sum_{s=\pm1}\theta(su\cdot p)(f_{s,R} - f_{s,L}) \end{aligned} \tag{18}$$

$$f_{s,\chi} = \frac{2}{(2\pi)^3}\frac{1}{e^{s(u\cdot p-\mu_\chi)/T}+1}, \qquad (\chi = R, L). \tag{19}$$

From now on, we focus on the rotating frame, and the curvature R_{ab} is identically zero. Using the equations (12-14), we can obtain $\mathcal{V}^a$ and $\mathcal{A}^a$ to the first order

$$\begin{aligned} \mathcal{V}^a &= p^a\delta(p^2)V^0 + \frac{1}{2}p_k(u^ks^a - u^as^k)A_0'\delta(p^2) - \frac{1}{4p^2}\epsilon^{abcd}p_bp_n\omega_c^{[mn]}\eta_{md}A_0\delta(p^2), \\ \mathcal{A}^a &= p^a\delta(p^2)A^0 + \frac{1}{2}p_k(u^ks^a - u^as^k)V_0'\delta(p^2) - \frac{1}{4p^2}\epsilon^{abcd}p_bp_n\omega_c^{[mn]}\eta_{md}V_0\delta(p^2). \end{aligned} \tag{20}$$

Then we have the vector and axial-vector currents

$$\begin{aligned} j^a &= \int d^4p\mathcal{V}^a = nu^a + \xi s^a + \zeta\theta^a, \\ j_5^a &= \int d^4p\mathcal{A}^a = n_5u^a + \xi_5s^a + \zeta_5\theta^a, \end{aligned} \tag{21}$$

where s^a is the fluid vorticity

$$s^a = \frac{1}{2}\epsilon^{abcd}u_b\partial_cu_d, \qquad \theta^a = \epsilon^{abcd}u_bu_n\omega_c^{[mn]}\eta_{md}. \tag{22}$$

The coefficients

$$\begin{aligned} n &= \int d^4pp_0V_0\delta(p^2), \\ n_5 &= \int d^4pp_0A_0\delta(p^2), \\ \xi &= \frac{1}{2}\int d^4pp_0A_0'\delta(p^2), \\ \xi_5 &= \frac{1}{2}\int d^4pp_0V_0'\delta(p^2), \\ \zeta &= -\frac{1}{4}\int d^4p\frac{p_0^2}{p^2}A_0\delta(p^2), \\ \zeta_5 &= -\frac{1}{4}\int d^4p\frac{p_0^2}{p^2}V_0\delta(p^2). \end{aligned} \tag{23}$$

where

$$A_0' = \frac{\partial A_0}{\partial(u\cdot p)}, \qquad V_0' = \frac{\partial V_0}{\partial(u\cdot p)}. \tag{24}$$

Conclusion

Our calculation shows that the vector and axial vector currents are in direct proportion to the fluid vorticity and the angular velocity of the rotating system. Actually, using the method of integration by parts one can find that the coefficients ξ and ζ are same, so do the ξ_5 and ζ_5. It indicates the profound relationship between the vorticity and the rotation of the system.

References

[1] Calzetta E, Habib S, Hu B L. Quantum kinetic field theory in curved spacetime: Covariant Wigner function and Liouville-Vlasov equations.[J]. Physical Review D Particles And Fields, 1988, 37(10):2901-2919.

[2] Gao J H, Liang Z T, Pu S, et al. Chiral anomaly and local polarization effect from the quantum kinetic approach[J]. Physical Review Letters, 2012, 109(23):232301.