Stokes equation in a toy CD hovercraft

2011 Eur. J. Phys. 32 89

(http://iopscience.iop.org/0143-0807/32/1/008)

Eur. J. Phys. **32** (2011) 89–99 doi:10.1088/0143-0807/32/1/008

Stokes equation in a toy CD hovercraft

Charles de Izarra and Grégoire de Izarra

Groupe de Recherche sur l'Énergetique des Milieux Ionisés, UMR6606 Université d'Orléans—CNRS, Faculté des Sciences, Site de Bourges, rue Gaston Berger, BP 4043, 18028 Bourges Cedex, France

E-mail: Charles.De_Izarra@univ-orleans.fr

Received 25 July 2010, in final form 20 September 2010
Published 17 November 2010
Online at stacks.iop.org/EJP/32/89

Abstract

This paper deals with the study of a toy CD hovercraft used in the fluid mechanics course for undergraduate students to illustrate the lubrication theory described by the Stokes equation. An experimental characterization of the toy hovercraft (measurements of the air flow value, of the pressure in the balloon and of the thickness of the air film under the hovercraft) allows us to evaluate a reduced Reynolds number R^*. Since $R^* < 1$, it is possible to simplify the Navier–Stokes equation that is reduced to the Stokes equation, on the basis of the lubrication theory. The pressure gradient in the air flow is calculated, allowing us to establish the lifting force applied on the toy hovercraft. In addition, these results are applied to a larger scale hovercraft.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

A lesson on fluid mechanics for undergraduate students generally begins with fluid statics, followed by the study of ideal fluid dynamics described by the Euler equation. Introduction of the fluid viscosity leads to the Navier–Stokes equation that can be analytically solved only in a very reduced number of cases [1]. When the inertial force is negligible in front of the viscous force, the Navier–Stokes equation describes fluid dynamics *at a low Reynolds number*, which is of great importance in many fields of sciences, from lubrication theory to the study of Paramecia motion [2]. A possible course demonstration that illustrates the fluid effects at low Reynolds number is obtained by dropping a sheet of paper on a table: the thin air layer between the table and the paper sheet produces an important pressure variation and a lifting force; the paper sheet glides on the table. In this paper, we propose a small science toy easily built that may be used for a demonstration course or for a fluid mechanics exercise.

2. Experiment: experimental results

2.1. Building a toy CD hovercraft

The construction of a toy CD hovercraft needs a CD, a medium-sized balloon and a sports bottle lid that should be opened by pulling the nozzle and closed by performing the same

Figure 1. Construction of a toy CD hovercraft with a CD, a sports bottle lid and a balloon.

action (figure 1). The bottle lid is glued over the hole of the CD, and the balloon is attached to the opening of the lid. To operate the hovercraft, it is necessary to open the sports bottle lid, to blow air into the balloon through the CD hole, and to close the lid so that the air cannot escape. The CD is placed on a smooth surface, the lid is carefully opened with a small aperture and the hovercraft glides slowly on the surface.

2.2. Assumptions and measurements

Different measurements were performed on the toy CD hovercraft for which the balloon was chosen spherical. The total mass of the system filled with air is $m = 21.45$ g. Using a pressure sensor, we have checked that the overpressure P_B in the balloon was nearly constant during the time when the balloon empties: $P_B = (1000 \pm 100)$ Pa.

The volume V of air contained in the balloon at pressure $P_B + P_A$, P_A being the atmospheric pressure, was determined by measuring the diameter of the spherical balloon. The time needed to completely empty a balloon through the small aperture of the lid is about 15 s; consequently, the process can be considered isothermal. The volume V' of air contained in the balloon evaluated for the atmospheric pressure P_A is given by $V' = (P_A + P_B)V/P_A$. Since $P_B = 0.01P_A$, it is clear that V is very close to V', and we can take $V = V'$.

Between the cylinder of radius $R_0 = 6$ mm located in the central part of the CD and the cylinder of radius $R_1 = 60$ mm, we can consider an air flow with a cylindrical symmetry located in a thin film of height h (figures 2 and 3). The pressure value P_0 in the central part of the system (cylinder of radius R_0) is lower than the pressure inside the balloon; the value of P_0 depends on the flow value, and will be calculated later.

The thickness h of the air film between the CD surface and the table was measured using two methods. The first method uses sheets of adhesive papers whose thicknesses were

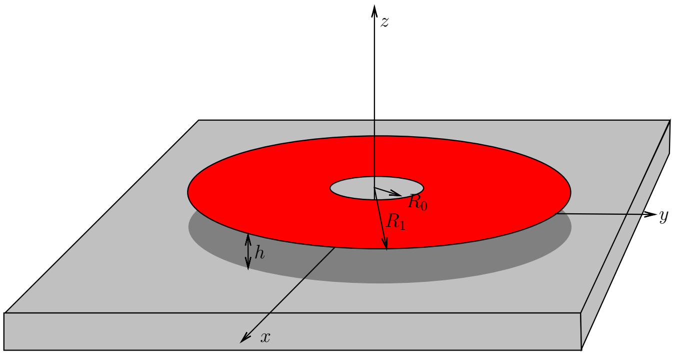

Figure 2. The toy CD hovercraft on a smooth surface.

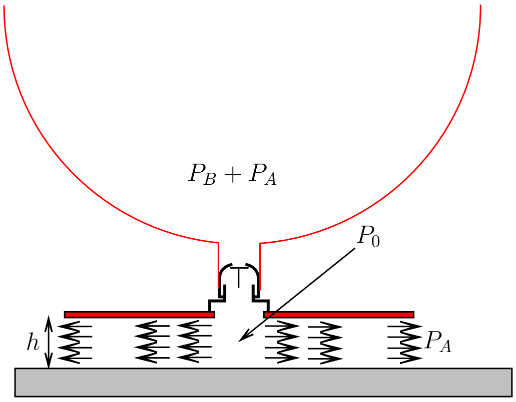

Figure 3. Air flow between the CD and the smooth surface (lateral view along a diameter).

determined using a micrometer screw gauge, successively placed on a sheet of paper put on the table. When the toy CD hovercraft is stopped by the paper, we can consider that the thickness of the paper is higher than h. This simple method indicates an h value located between 0.4 and 0.6 mm. A second method uses a non-contact optical displacement measuring system (OPTONCDT LD 1605-20 from MICRO-EPSILON); we obtain $h = (0.55 \pm 0.05)$ mm.

To measure the volumetric air flow Q, we have applied the following procedure that needs a chronometer and two operators. First, the spherical balloon is inflated, and its circumference is measured using a flexible tape measure. The circumference of the balloon is used to calculate the volume of air V it contains. The first operator places the hovercraft on a flat surface and carefully opens the lid. Simultaneously, the second operator starts a timer and stops when the balloon is empty to measure the time τ. The volumetric air flow is given by $Q = V/\tau$. We obtain the value 0.3×10^{-3} m^3 s^{-1}. Considering the flow incompressible (justified because the Mach number is lower than unity), we have $Q = v_0 S_0$ where v_0 is a mean air velocity value at the radius R_0 and $S_0 = 2\pi h R_0$ is the lateral surface of the cylinder of radius R_0 and height h. Numerically, $v_0 = 16$ m s^{-1}. The same approach for the radius R_1 indicates an air velocity value $v_1 = 1.6$ m s^{-1}. For any value of the radius r, r being the radial coordinate, the mean velocity is $v_m = Q/2\pi hr$, and a possible mean value U of the air velocity evaluated over the total surface of the CD is given by

$$U\pi\left(R_1^2 - R_0^2\right) = \int_{R_0}^{R_1} v_m 2\pi r\,\mathrm{d}r = \int_{R_0}^{R_1} \frac{Q}{2\pi rh} 2\pi r\,\mathrm{d}r$$

that reduces to

$$U = \frac{Q}{\pi h(R_1 + R_0)}.$$

Numerically, we obtain $U = 2.6\ \mathrm{m\,s^{-1}}$.

3. Theoretical approach

3.1. From the Navier–Stokes equation to the Stokes equation

Let us consider the Navier–Stokes equation for an incompressible fluid of specific mass ρ and dynamical viscosity μ. We have, neglecting the external forces such as the weight,

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}P + \mu\Delta\vec{v}.$$

In the case of a permanent flow, $\partial\vec{v}/\partial t = \vec{0}$. The reduced Reynolds number R^* defined in [1] gives the ratio of inertia to viscous forces, in the case of a system having two very different typical lengths R_1 and h, with $R_1 \gg h$:

$$R^* = \frac{\text{inertia force}}{\text{viscous force}} = \frac{\rho(\vec{v}\cdot\vec{\nabla})\vec{v}}{\mu\Delta\vec{v}}.$$

The inertia force can be evaluated by considering the typical length R_1, while the viscous force is defined by considering the typical length h; using the averaged velocity U, we have, calling $\nu = \mu/\rho$ the kinematic viscosity,

$$R^* = \frac{\rho U^2/R_1}{\mu U/h^2} = \frac{UR_1}{\nu}\left(\frac{h}{R_1}\right)^2.$$

For air at ambient temperature and pressure, $\nu = 1.5\ 10^{-5}\ \mathrm{m^2\,s^{-1}}$, and the numerical value of the reduced Reynolds number R^* is 0.72, tending to zero as $h \to 0$. We can consider only the viscous and the pressure terms in the Navier–Stokes equation, which reduces to the Stokes equation, at the basis of lubrication theory [3].

3.2. Pressure distribution

The starting point is the Stokes equation

$$-\vec{\nabla}P + \mu\Delta\vec{v} = \vec{0} \tag{1}$$

and the continuity equation for an incompressible flow

$$\vec{\nabla}\cdot\vec{v} = 0. \tag{2}$$

Taking the cylindrical coordinates (r, φ, z), the velocity vector is $\vec{v} = v_r\vec{e}_r + v_\varphi\vec{e}_\varphi + v_z\vec{k}$, where $(\vec{e}_r, \vec{e}_\varphi, \vec{k})$ is the orthonormal local basis of the cylindrical coordinate system. In our case, it is clear that the celerity components v_φ and v_z are null. The radial component v_r only depends on the variables r and z. Using the definition of the Laplacian of a vector and the expression of the gradient in cylindrical coordinates [4], equation (1) becomes

$$\frac{\partial P}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial P}{\partial\varphi}\vec{e}_\varphi + \frac{\partial P}{\partial z}\vec{k} = \mu\left(\Delta v_r - \frac{v_r}{r^2}\right)\vec{e}_r, \tag{3}$$

with

$$\Delta v_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{\partial^2 v_r}{\partial z^2} \quad \Rightarrow \quad \Delta v_r = \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2}. \tag{4}$$

From equation (2), we have

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} = 0 \quad \Rightarrow \quad \frac{\partial v_r}{\partial r} = -\frac{v_r}{r}. \tag{5}$$

The operator $\partial/\partial r$ applied to equation (5) gives

$$\frac{\partial^2 v_r}{\partial r^2} = -\frac{2}{r}\frac{\partial v_r}{\partial r} \quad \Rightarrow \quad \frac{\partial^2 v_r}{\partial r^2} = +\frac{2v_r}{r^2}. \tag{6}$$

Using the result given by equation (6), Δv_r (equation (4)) becomes

$$\Delta v_r = \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2}. \tag{7}$$

Finally, equation (3) is reduced to

$$\frac{\partial P}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial P}{\partial \varphi}\vec{e}_\varphi + \frac{\partial P}{\partial z}\vec{k} = \mu\frac{\partial^2 v_r}{\partial z^2}\vec{e}_r. \tag{8}$$

A quick analysis of equation (8) indicates that the pressure does not depend on the angle φ, and is constant over the direction z, i.e. on the thickness of the air film. Integration of the radial component is straightforward:

$$v_r = \frac{1}{2\mu}\frac{\partial P}{\partial r}z^2 + Az + B \tag{9}$$

with the integration constants A and B found with the limit conditions $v_r(z = 0) = 0$ and $v_r(z = h) = 0$. We obtain the parabolic shape

$$v_r = \frac{1}{2\mu}\frac{\partial P}{\partial r}(z^2 - hz). \tag{10}$$

3.3. Lifting force

Knowing the velocity $v_r(z)$ from equation (10), it is possible to compute the volumetric air flow Q:

$$Q = \iint \vec{v}_r \cdot \mathrm{d}\vec{S},$$

with $\mathrm{d}\vec{S}$ the elementary lateral surface of a cylinder of radius r and height h. The calculations give

$$Q = -\frac{\pi r}{6\mu}\frac{\partial P}{\partial r}h^3. \tag{11}$$

Integration of the pressure gradient

$$\frac{\partial P}{\partial r} = -\frac{6\mu Q}{\pi h^3}\frac{1}{r}, \tag{12}$$

taking into account that the pressure $P = P_A$ for $r = R_1$, leads to the pressure distribution $P(r)$ plotted in figure 4:

$$P(r) = P_A + \frac{6\mu Q}{\pi h^3}\ln\left(\frac{R_1}{r}\right). \tag{13}$$

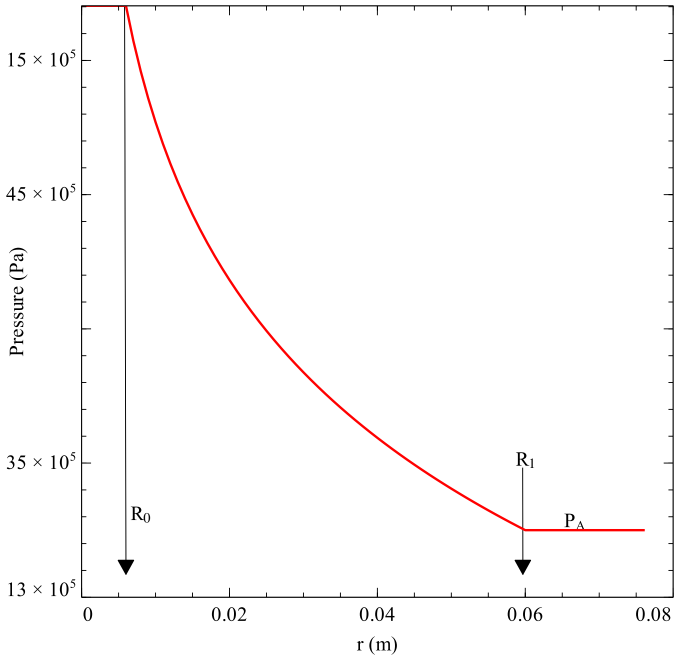

Figure 4. Plot of the pressure in the air film under the toy CD hovercraft.

The lifting force F is obtained by integrating the overpressure $\delta P(r) = P(r) - P_A$ over all the surface of the CD; let us underline that the overpressure in the central area of radius R_0 is constant, equal to $\frac{6\mu Q}{\pi h^3} \ln\left(\frac{R_1}{R_0}\right)$. We have

$$F = \int_{R_0}^{R_1} \delta P(r) 2\pi r \, \mathrm{d}r + \pi R_0^2 \delta P(R_0)$$

or

$$F = \int_{R_0}^{R_1} \frac{6\mu Q}{\pi h^3} \ln\left(\frac{R_1}{r}\right) 2\pi r \, \mathrm{d}r + \pi R_0^2 \delta P(R_0).$$

$$F = \frac{12\mu Q}{h^3} \left(\int_{R_0}^{R_1} r \ln R_1 \, \mathrm{d}r - \int_{R_0}^{R_1} r \ln r \, \mathrm{d}r \right) + \pi R_0^2 \frac{6\mu Q}{\pi h^3} \ln\left(\frac{R_1}{R_0}\right).$$

Integration of the second integral is made by parts; finally, we obtain

$$F = \frac{3\mu Q}{h^3} \left(R_1^2 - R_0^2\right). \tag{14}$$

The numerical evaluation of the force F with the experimental results proposed in subsection 2.2 and $\mu = 1.85 \times 10^{-5}$ kg m^{-1} s^{-1} is $F = 0.35$ N. This value must be compared to the weight of the toy CD hovercraft (0.210 N); taking into account the experimental errors (30% of relative error on h measurement), it is clear that the lifting force is of the same magnitude as the weight of the system, and allows it to glide on a table.

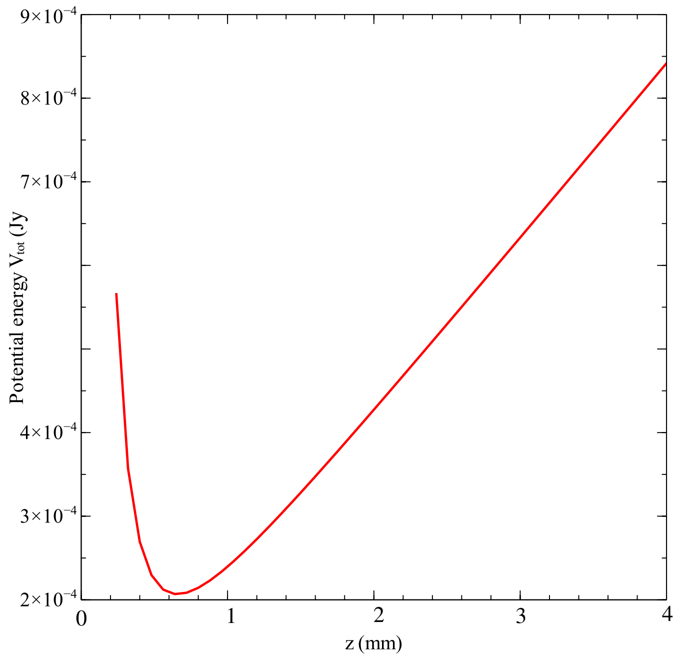

Figure 5. Plot of the potential energy curve of the toy CD hovercraft.

3.4. Potential energy curve and stability

Writing the lifting force as a function of z $\vec{F} = C/z^3\vec{k}$, with $C = 3\mu Q\left(R_1^2 - R_0^2\right)$, the corresponding potential energy function V ($\vec{F} = -\vec{\nabla}V$) is, choosing $V \to 0$ for $z \to \infty$,

$$V = \frac{C}{z^2}. \tag{15}$$

Taking into account the potential energy for the weight $V_w = mgz$, the total potential energy of the system is

$$V_{\text{tot}} = \frac{C}{z^2} + mgz. \tag{16}$$

The plot of V_{tot} as a function of z is presented in figure 5; the minimum of potential energy corresponds to a stable point of the system. Let us underline that vertical oscillations of the toy CD hovercraft are possible around this stable point.

4. Study of the loaded hovercraft

In this experimental part, we have put different known weights of the toy CD hovercraft, and we have measured the volumetric air flow Q. When the hovercraft glides on the table, the lifting force is equal to the weight of the loaded hovercraft. Experimental results are presented in figure 6. If we suppose that the mean velocity does not change radically for different loads (F varies from 0.2 N to 0.8 N), then the volumetric air flow Q is proportional to h, and F is proportional to Q/h^3; consequently, we can foresee the following law:

$$F = \frac{C}{Q^2}.$$

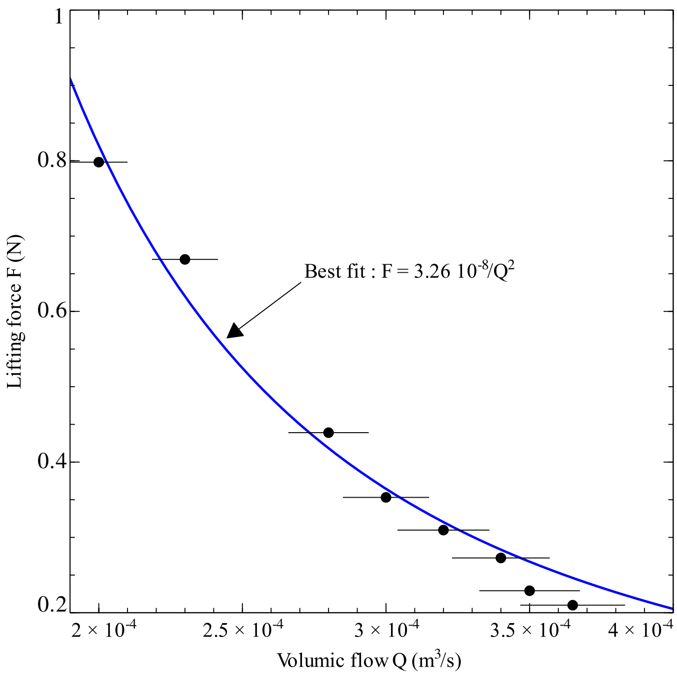

Figure 6. Plot of the lifting force value against the volume air flow.

The best fit given in figure 6 is in good agreement with experimental results, considering the experimental error mainly given by the measurements of the volume air flow.

5. Test of a bigger hovercraft

In this section, we present experimental tests obtained on a larger hovercraft, for which we give the construction procedure.

5.1. Construction

According to [5, 6], we have built a larger toy hovercraft with a leaf blower. The basis of the larger hovercraft is a disc of plywood (thickness 1 cm) of radius $R_1 = 0.6$ m, and a circular plastic sheet (chose a sheet of heavy plastic) of radius larger than the plywood disc (radius of about 0.8 m) (figures 7(a) and (b)).

Half way between the centre of the disk and the edge, a hole is made in the plywood that exactly fits the end of the leaf blower. The plywood disk is laid on the centre of the large plastic sheet. The edges of the sheet are folded up over the plywood, and fixed to the top of the plywood disc using a staple gun. Adhesive tape is used to tape the edge of the plastic down to make it look good (figures 7(b) and (c)).

On the bottom side of the hovercraft, the plastic sheet must be fixed to the centre of the plywood disc by using a rigid plastic disc (radius of about 5 cm) easily found on a coffee can lid. The rigid plastic disc is fixed on the plywood disc using a staple gun (figure 7(d)).

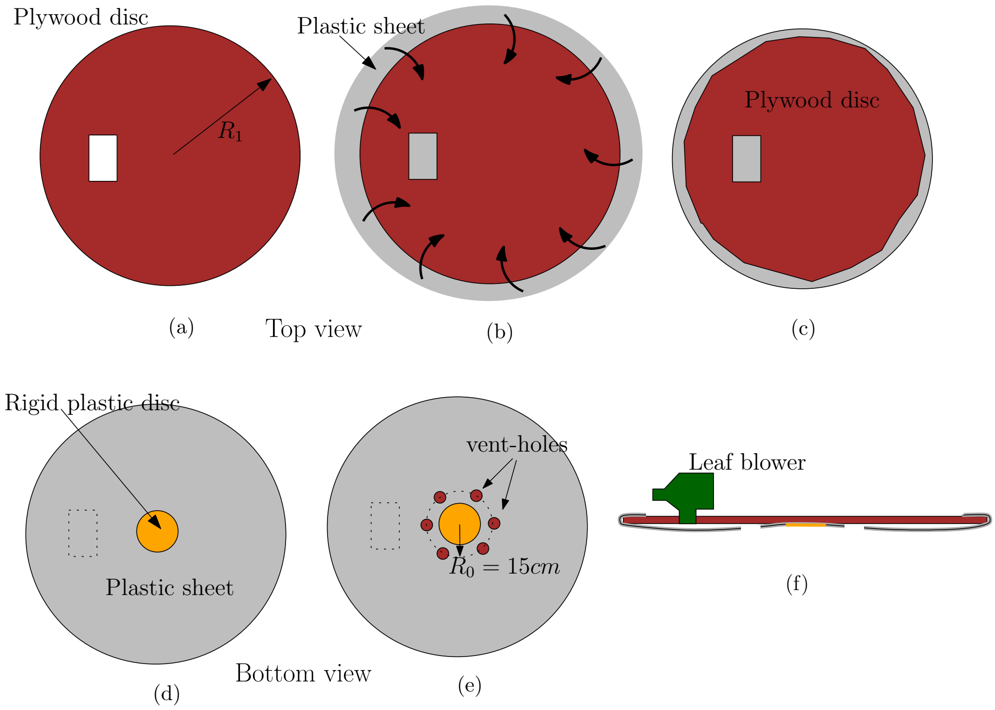

Figure 7. Larger hovercraft construction.

Figure 8. Larger hovercraft in operation.

With a razor knife, six vent-holes (radius of 4 cm) are cut in the plastic sheet (figure 7(e)). They are placed on a circle of radius $R_0 = 0.15$ m. The experiment indicates that it is necessary to reinforce the thin necks of plastic between the holes using adhesive tape.

Flip the hovercraft over so that the plastic sheet is on the bottom and place it on a smooth floor. Put the end of the leaf blower into the hole and turn it on. The plastic on the bottom should inflate; the hovercraft lifts up slightly and starts gliding around (figure 8).

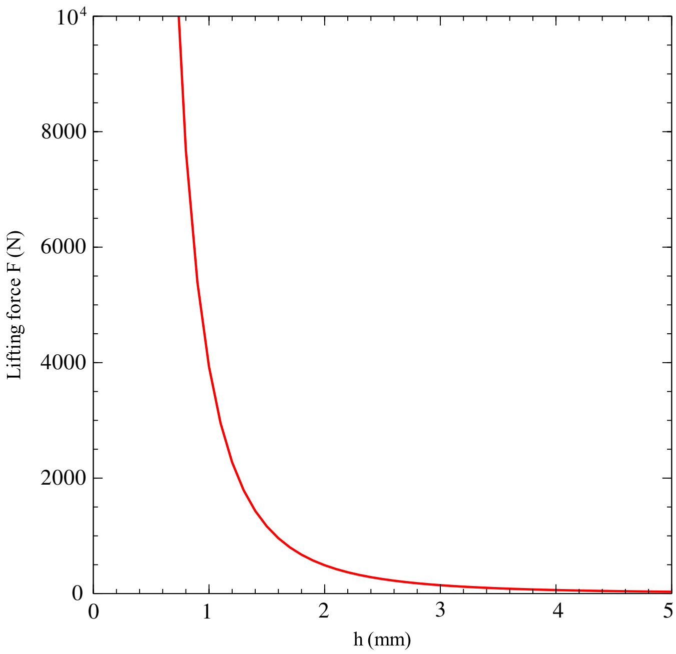

Figure 9. Plot of the lifting force value F (expression (14)) against the air film thickness h for the larger hovercraft.

5.2. Results

The theoretical expressions found for the small CD hovercraft (lifting force) are valid for the larger hovercraft. The maximum volumetric air flow indicated by the leaf blower constructor is $Q = 0.21$ m^3 s^{-1}, and the radii defined in the previous sections are $R_1 = 0.6$ m and $R_0 = 0.15$ m. The value of the thickness of the air film h between the ground surface and the plastic sheet is difficult to evaluate with accuracy, and is located in the range [1 mm, 2mm]. However, assuming that Q is constant, the plot of the theoretical value of the lifting force F (expression (14)) as a function of h (figure 9) indicates rather high values of the lifting force F. It was possible for us to lift four persons (mass $\approx$ 350 kg) on a linoleum surface, corresponding to a lifting force of 3433 N.

6. Conclusion

This experimental study of the toy CD hovercraft is rich enough to illustrate the analysis of the Navier–Stokes equation as a function of the Reynolds number value, in the case of incompressible flows. It is used during the fluid mechanics course, and can lead to completely analytical problems, difficult to find in the topics of fluid mechanics.

Acknowledgment

The authors wish to thank the anonymous reviewers for their constructive comments.

References

[1] Schlichting H 1968 *Boundary-Layer Theory* (New York: McGraw-Hill)
[2] Purcell E 1977 *Am. J. Phys* **45** 11
[3] Sen M 1993 *Appl. Math. modelling* **17** 226–35
[4] Huba J 2009 *NRL Plasma Formulary 2009*
[5] Burchill D 2003 *Phys. Educ.* **38** 490
[6] Altshuler K 1989 *Phys. Teacher* **27** 615–7