Journal of the Physical Society of Japan
Vol. 54, No. 7, July, 1985, pp. 2462–2469

Vibration of a Flattened Drop. II. Normal Mode Analysis

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(Received December 7, 1984)

A hydrodynamical model is developed to analyse vibration of a liquid drop on a horizontal bed. The drop has a flattened shape by the gravity and the surface tension. The fluid is assumed inviscid and to move nearly horizontally, so that the shallow water theory is applied, and a set of governing equations for the vibration is derived. It is applied to a small-amplitude oscillation and a normal mode solution is obtained, where the drop shows a standing wave along its periphery. The predicted vibration frequency agrees with the experiment by the present authors.

§1. Introduction

It is well known that liquids in the air, such as a spherical drop or a circular column, show vibrations under the actions of the surface tension and the inertia. Dynamics of these motions have long been a subject of interest, as reviewed by Levich *et al.*[1]

Angular frequencies of normal modes of these vibrations are first derived by Rayleigh in the following forms:[2,3]

$$\omega_{\mathrm{d}}^2=\frac{(n-1)n(n+2)\sigma}{r_0^3\rho} \quad \text{for a spherical drop,} \tag{1}$$

$$\omega_{\mathrm{c}}^2=\frac{(n-1)n(n+1)\sigma}{r_0^3\rho} \quad \text{for a circular column,} \tag{2}$$

where n is a mode number and stands for a number of vertices appearing on the liquid surface during vibration, r_0 is an average radius of the drop or the column, and σ, ρ are the surface tension coefficient and the density of the liquid, respectively.

Formula (2) agrees with the experiment by Rayleigh where a liquid jet was ejected into the air.[3] On the other hand, experiments to observe drop vibration were made by applying external forces to cancel the gravity, such as a buoyancy in an immiscible liquid with the same density as the drop[4-6] or a drag force on a drop in a vertical wind tunnel.[7]

Recently, the present authors used a method completely different from above to observe a drop vibration.[8] A liquefied-gas drop was put on a horizontal bed with the room temperature, where the drop began vibrating naturally by an energy input from rapid evaporation while suffering no friction from the bed owing to a thin vapor layer under the drop. Average shape of the drop was a circular disk and its plane view showed a characteristic oscillation, where the fluid on the periphery of the disk moved nearly horizontally and a standing wave appeared along the periphery. In this experiment the vibrational frequency was measured, which showed the $r_0^{3/2}$ dependence as in eqs. (1) and (2). On the other hand, the measured dependence on n revealed a peculiar nature, i.e. there was a clear distinction between even and odd numbers of n. Another interesting nature of this motion was an occurrence of sudden transition to another mode with smaller n as the drop volume was reduced through evaporation. Moreover, by the help of a simple analysis, it was concluded that a combination of material constants $(\sigma/\rho)^{1/2}$, estimated from experimental data is about 10% smaller than that in the thermal equilibrium at the boiling temperature.

As is seen from these results, the vibration of liquefied-gas drop contains a lot of interesting theoretical problems worth investigating, such as how the vibration is sustained, how the transition is provoked, whether the material constants of evaporating liquid are the same as those in the thermal equilibrium, etc. It is also worth noting that this motion is looked upon as a dynamical pattern formation under the inequilibrium condition, one of the most attractive topics of today. Moreover, the problem of mechanism of the mode transition mentioned above will afford a new field of research in the fundamental theoretical physics. Series of the present works is motivated by the interest in the phenomenon and the necessity to obtain physical understanding of these problems.

Since the problem is basically a hydrodynamical one, it is appropriate to begin with a simplified model for vibration of a flattened drop without complicating factors, such as interaction with outer gas or temperature variation on the liquid surface. Nevertheless, analysis is associated with several difficulties, since the fluid in vibration makes a complicated three-dimensional motion by the action of the gravity and the surface tension. By this reason also, the Rayleigh's result for a sphere or a circular column will give no true understanding of the phenomenon.

It is the purpose of this paper to propose a set of equations to analyse this complicated motion of the flattened drop, and, as a check of validity of the theory, to predict frequency of small-amplitude vibration. This is the first step for physical understanding of the phenomena, and extension of the theory to include further interesting factors, such as the finite amplitude, the vibration-sustaining mechanism and the mode transition, will be treated in future papers.

In the following sections, a set of governing equations are derived on several assumptions (§2), the equations are linearized and its normal mode solution is obtained (§3), and the result is compared with the experiment by the present authors (§4).

§2. Governing Equations

2.1 Assumptions

Even if we confine ourselves to a drop without evaporation, regorous treatment is still difficult, because the drop has a complex distribution of curvature on its surface. For a drop at rest on a horizontal bed, an equilibrium shape is obtained theoretically by Kubo.[9)] The phenomenon in question is an oscillation of the drop surface around this equilibrium state. Analysis of such a motion, however, will require a very complicated mathematics. Here, for the sake of simplicity, several assumptions are made on the geometry of the drop and the dynamics of vibration as listed below.

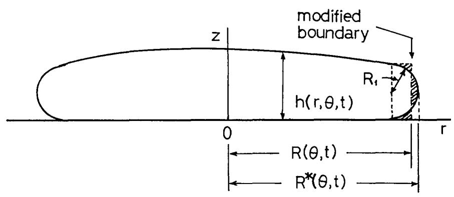

Fig. 1. Assumed geometry of a flattened drop (side view) and definitions of variables. Hatched regions show how the modified boundary is defined.

(i) The drop is looked upon as a water layer and the shallow water theory is applied to the fluid motion. The layer is prevented from spreading by the surface tension at the peripheral region (see Fig. 1). The peripheral region has a semi-circle vertical cross section. The drop at rest has a thickness h_0, and its plane view is a circle with radius r_0, where $r_0 \gg h_0$.

(ii) The vibration of the drop is a surface wave on the shallow water, and the peripheral region has a role to give boundary conditions to the fluid motion through the surface tension. The inertia effect of the peripheral region is considered by introducing a hypothetical boundary, say a modified boundary. The modified boundary is defined as a vertical wall at a position, where the area of the vertical cross section remains unchanged after replacing the true boundary by this modified boundary (see Fig. 1). The shallow water layer is assumed to extend to the modified boundary.

(iii) Lower surface of the shallow water region remains horizontal and suffers no friction from the bed.

(iv) Flow in the drop is incompressible and inviscid, hence the potential flow theory is applied.

(v) Effect of evaporation is neglected, hence all material parameters and the drop volume are constant during vibration.

Brief comments are given about these assumptions. Assumptions (i), (ii) and (iii) do not contradict with observation except the introduction of the modified boundary. As for (iv), an estimation of the Reynolds number is necessary. In the experiment the oscillation amplitude at the periphery was about 0.1 cm/s and typical frequency was 30 Hz for a drop of diameter 5 mm. The Reynolds number based on these values has a magnitude of $O(10^2)$. Therefore, the assumption (iv) is acceptable even for a small drop considered here. The constancy of the drop volume in (v) will be valid for analysis of vibration frequency, because the volume changed little during one period, while the constancies of material parameters are for simplification of analysis.

2.2 Derivation of governing equations

A polar coordinate system (r, θ) is introduced in the horizontal plane. The contour in the plane view and the modified boundary are expressed as $r=R^*(\theta, t)$ and $r=R(\theta, t)$, respectively. The depth and the two-dimensional fluid velocity in the shallow water layer are denoted by $h(r, \theta, t)$ and $\boldsymbol{u}(r, \theta, t)=(u_r, u_\theta)$ respectively, where t is the time. Relation between R^* and R then becomes, from the definition of the modified boundary,

$$R^*(\theta, t)= R(\theta, t)+\beta h(R, \theta, t), \quad \left(\beta=\frac{1}{2}-\frac{\pi}{8}=0.107\right), \tag{3}$$

where the value of β is obtained from the consideration of area mentioned in the assumption (ii).

Basic equations for the shallow water region are the continuity equation and the momentum equation as follows:

$$\frac{\partial h}{\partial t}+\frac{1}{r}\frac{\partial(rhu_r)}{\partial r}+\frac{1}{r}\frac{\partial(hu_\theta)}{\partial\theta}=0, \tag{4}$$

$$\frac{\partial}{\partial t}(\rho h\boldsymbol{u})+\frac{\partial}{\partial r}(u_r\rho h\boldsymbol{u})+\frac{1}{r}\frac{\partial}{\partial\theta}(u_\theta\rho h\boldsymbol{u})=-\text{grad } P, \tag{5}$$

where P is an integration of the fluid pressure p in the vertical direction. This integrated pressure P comes both from the gravity and the surface tension at the upper surface, so that it is expressed as

$$P=\int_0^h p\, \mathrm{d}z=\frac{1}{2}\rho gh^2-\sigma h\Delta h, \tag{6}$$

where z is the vertical coordinate and Δ is the two-dimensional Laplacian $\partial^2/\partial x^2+\partial^2/\partial y^2$.

Two boundary conditions are posed at $r=R$. One is a pressure boundary condition. From assumption (ii) the integrated pressure P at the modified boundary balances with that due to the surface tension at the peripheral region, i.e.

$$P=h\sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right), \tag{7}$$

where R_1 and R_2 are radii of curvature at the peripheral region. Because of the assumed geometry, R_1 is the radius of the semicircle in the vertical cross section, hence the half of the layer depth at the boundary, while R_2 is a radius of curvature of the contour in the plane view of the drop (see Fig. 1).

The second is a kinematical boundary condition, i.e.

$$u_r=u_\theta\frac{1}{r}\frac{\partial R}{\partial\theta}+\frac{\partial R}{\partial t}, \quad \text{at } r=R. \tag{8}$$

2.3 Static state

Before going on to analysis of vibration, some estimations for a static drop are made. Let a modified boundary in the static state be at $r=R_0$ (=constant). Then, one has, from conditions (3) and (7),

$$\frac{1}{2}\rho gh_0^2=h_0\sigma\left(\frac{1}{h_0/2}+\frac{1}{R_0+\beta h_0}\right). \tag{9}$$

This relation suggests introduction of the following two non-dimensional parameters:

$$G=\frac{\rho gR_0^2}{\sigma}, \quad H=\frac{h_0}{R_0}. \tag{10}$$

The parameter G is the square of ratio of two time scales $T_0=(\rho R_0^3/\sigma)^{1/2}$ and $(R_0/g)^{1/2}$, indicating importance of the gravity relative to the surface tension in the shallow water region. The second is a geometrical parameter. In terms of these parameters eq. (9) is rewritten as

$$\frac{1}{2}GH^2=2+\frac{1}{H^{-1}+\beta}. \tag{11}$$

Assumption (i) means conditions $H\ll 1$ and $G\gg 1$. In the asymptotic case with $G\to\infty$ and

$H\to 0$, eq. (11) requires

$$GH^2=\frac{\rho g h_0^2}{\sigma}\to 4 \quad \text{and} \quad GH=O(H^{-1})\to\infty. \tag{12}$$

For drops of liquefied gases used in our experiment,[8] this asymptotic case corresponds to a depth $h_0=1.8$ mm, which does not differ much from the experimental value 1.5 mm. It should be noted here that eq. (12) agrees completely with the result by Kubo.[9]

§3. Small Amplitude Vibration

3.1 Linearization of equations

The set of equation obtained in the preceding section is applied to a normal mode vibration with small amplitude.

The radial coordinate and the time are normalized by R_0 and the time scale T_0 $(=(\rho R_0^3/\sigma)^{1/2})$, respectively, and same notations are used for normalized quantities. Dynamical variables are expressed as

$$h=R_0H(1+\tilde{h}(r,\theta,t)),\quad R=R_0(1+\tilde{R}(\theta,t)),\quad u_r=\frac{R_0}{T_0}\frac{\partial\tilde{\phi}}{\partial r},\quad u_\theta=\frac{R_0}{T_0}\frac{\partial\tilde{\phi}}{r\partial\theta}, \tag{13}$$

where $\tilde{\phi}$ is a velocity potential. Quantities $\tilde{h}$, $\tilde{R}$ and $\tilde{\phi}$ are nondimensional small perturbations, and all equations and conditions are linearized with respect to these quantities.

Substituting eq. (13) into eqs. (4) and (5), one has

$$\frac{\partial\tilde{h}}{\partial t}+\Delta\tilde{\phi}=0,\quad \text{grad}\left(\frac{\partial\tilde{\phi}}{\partial t}+GH\tilde{h}-H\Delta\tilde{h}\right)=0. \tag{14}$$

The second of these can be integrated once to give

$$\frac{\partial\tilde{\phi}}{\partial t}+GH\tilde{h}-H\Delta\tilde{h}=0, \tag{15}$$

where the integration constant is fixed to zero without loss of generality.

The pressure boundary condition (9) contains two radii of curvature. They are expressed in terms of perturbations as

$$\frac{1}{R_1}=\frac{1}{R_0}\frac{2}{H}(1-\tilde{h}),$$

$$\frac{1}{R_2}=\frac{1}{R_0(1+\beta H)}-\frac{1}{R_0(1+\beta H)^2}\left(\tilde{R}+\beta H\tilde{h}+\frac{\partial^2}{\partial\theta^2}(\tilde{R}+\beta H\tilde{h})\right),$$

whose derivations are given in Appendix. Then, from eqs. (6) and (7), we have

$$GH\tilde{h}-H\Delta\tilde{h}=\frac{\tilde{h}}{1+\beta H}-\frac{\tilde{R}+\beta H\tilde{h}+\frac{\partial^2}{\partial\theta^2}(\tilde{R}+\beta H\tilde{h})}{(1+\beta H)^2},\quad \text{at } r=1. \tag{16}$$

The kinematical condition (8) is linearized to

$$\frac{\partial\tilde{\phi}}{\partial r}=\frac{\partial\tilde{R}}{\partial t},\quad \text{at } r=1. \tag{17}$$

Equations (14)–(17) constitute equations and boundary conditions for small-amplitude vibration of a flattened drop.

3.2 Normal mode solution

For a vibration with n vertices in the peripheral region, perturbations are written as

$$(\tilde{\phi},\tilde{h},\tilde{R})=(\hat{\phi}(r),i\hat{h}(r),i\hat{R})\exp(in\theta+i\Omega t), \tag{18}$$

where $\hat{\phi}$, $\hat{h}$, and $\hat{R}$ are seen to be real quantities from inspection of the governing equations. The problem is to obtain the value of the normalized angular frequency Ω. Frequency with dimension is denoted by ω for later use.

By substituting above expressions into eqs. (14)–(17) and eliminating $\hat{h}$ and $\hat{R}$, we have the following equation and boundary condition for $\hat{\phi}$:

$$(1-G^{-1}\varDelta_n)\varDelta_n\hat{\phi}+w^2\hat{\phi}=0, \tag{19}$$

$$(1-G^{-1}\varDelta_n)\varDelta_n\hat{\phi}=\frac{\varDelta_n\hat{\phi}}{GH(1+\beta H)}-\frac{(n^2-1)(\hat{\phi}'-\beta H\varDelta_n\hat{\phi})}{GH(1+\beta H)^2},\quad \text{at } r=1, \tag{20}$$

where

$$\varDelta_n=\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r}-\frac{n^2}{r^2},\quad w^2=\frac{\Omega^2}{GH},$$

and the prime denotes the derivative with respect to r.

Equation (19) is a differential equation of the fourth order and its solution is quite complicated. However, since we are considering the case $H \ll 1$, we can simplify above equations by neglecting terms of $O(H^2)$. Then, terms with coefficient G $(=O(H^2))$ are neglected, which means that we neglect the surface tension in the shallow water region (the surface tension is still working as a boundary condition at $r=1$). Equations (19) and (20) are then reduced to

$$(\varDelta_n+w^2)\hat{\phi}=0, \tag{21}$$

$$(n^2-1)\hat{\phi}'=w^2\{(1+\beta H)^2GH-(1+\beta H)-(n^2-1)\beta H\}\hat{\phi},\quad \text{at } r=1, \tag{22}$$

where eq. (21) is used in deriving eq. (22).

Solution of eq. (21) free from singularity at the origin is the Bessel function of the first kind, i.e.

$$\hat{\phi}=AJ_n(wr), \tag{23}$$

where A is an arbitrary constant. By substituting this into the condition (22), we obtain the final equation for the angular frequency

$$\frac{J_n'(w)}{J_n(w)}=\frac{GHw\{(1+2\beta H)-(GH)^{-1}+O(H^2)\}}{n^2-1}, \tag{24}$$

where the prime denotes the derivative.

Equation (24) has an infinite number of roots, but we are interested in the smallest root from the following reason. Larger roots correspond to waves with smaller wave lengths in the radial direction. However, in our experiment,[8] the simple wave without node within $0<r<R$ predominated, thus supporting this smallest root.

Approximate solution of eq. (24) is obtained by the use of the the asymptotic formula of Bessel function,

$$\frac{J_n'(w)}{J_n(w)}=\frac{n}{w}-\frac{J_{n+1}(w)}{J_n(w)}=\frac{n}{w}-\frac{w}{2(n+1)}+O(w^3). \tag{25}$$

In the limit $GH\to\infty$ (hence $H\to 0$ and $w\to 0$), eq. (24) is reduced, by the formula (25), to

$$\Omega^2=n(n^2-1). \tag{26}$$

It is interesting to note that this result coinsides with eq. (2) for a liquid column, whose reason would be the following. Though the curvature $1/R_1$ at the peripheral region produces a strong surface tension effect, its timewise variation during vibration is not so large, because the fluid is moving nearly horizontally. On the other hand, the variation of the second curvature $1/R_2$, which is common with the column, is effective for the dynamics of vibration.

For finite values of GH and H, correction to eq. (26) can be obtained by taking the second term of eq. (25), as follows:

$$\Omega^2=n(n^2-1)\Big/\left\{(1+2\beta H)+\frac{n-3}{2GH}\right\}, \tag{27}$$

where terms of $O(H^2)$ are neglected in the denominator. Note that $w=O(H^{1/2})$, and within the present approximation we should take only up to the second term in the expansion (25). This result is shown Fig. 2 by solid lines for $2\leqq n\leqq 8$, where H is varied as an independent parameter and GH is obtained from eq. (11). Asymptotic values obtained from eq. (26) are indicated in the same figure by horizontal lines.

It is concluded from Fig. 2 that the effect of finite H is to reduce the frequency from values given by eq. (26) and that this effect is stronger for larger n.

The theoretical curves could not be extended to the region with $H\sim 1$; they are shown in

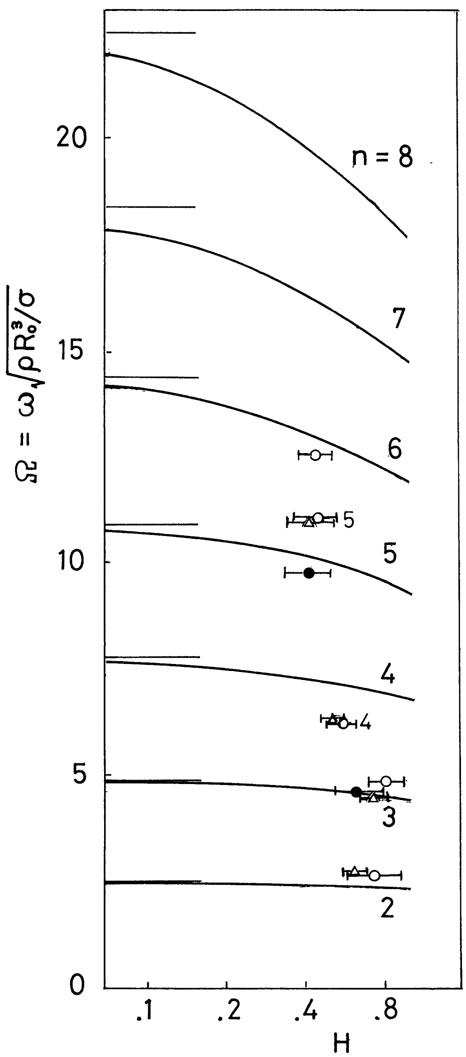

Fig. 2. Theoretical values of vibration frequency normalized by $(\rho R_0^3/\sigma)^{1/2}$. Thick curves are from the second approximation (see eq. (27)) in solving the eigenvalue equation. Thin horizontal lines are from Rayleigh's formula (2). ●, ○, △: respectively, experimental values for liquid N_2, O_2, Ar by the present authors, where $(\sigma/\rho)^{1/2}=2.9$, 3.2, 2.9 are used in normalizing the data.

Fig. 2 to show qualitatively the effect of the parameter H. They are necessary also from the reason, as will be explained in §4, that experimental values for very small H are not available.

§4. Comparison with Experiment and Discussion

Experimental values of $\Omega=\omega(\rho R_0^3/\sigma)^{1/2}$ by Adachi & Takaki[8)] are plotted in Fig. 2. The horizontal bars at data come from variation of drop radius due to evaporation during the frequency measurement, for example, the $n=6$ vibration of liquid O_2 appeared for $0.38 \leqq H \leqq 0.52$ while experimental value of Ω remained nearly constant. In normalizing experimental data, values of $(\sigma/\rho)^{1/2}$ must be specified. It is not certain whether standard material constants at thermal equilibrium can be applied. Here, in order that the experimental values fit to the theoretical curves, $(\sigma/\rho)^{1/2}$ is assumed to be 2.9, 3.2 and 2.9 for the liquid N_2, O_2 and Ar, instead of standard values 3.4, 3.5 and 3.1, respectively. Then, the agreement of the present theory with experiment is satisfactory.

Normalized frequencies both from Rayleigh's formulae and the present theory are plotted against n, as shown in Fig. 3. Since the Rayleigh's theory does not contain the concept of modified boundary, the radius r_0 in the plane view is used in normalizing frequency. Therefore, values of Ω from the present theory are multiplied by $(r_0/R_0)^{3/2}=(1+\beta H)^{3/2}$. Since

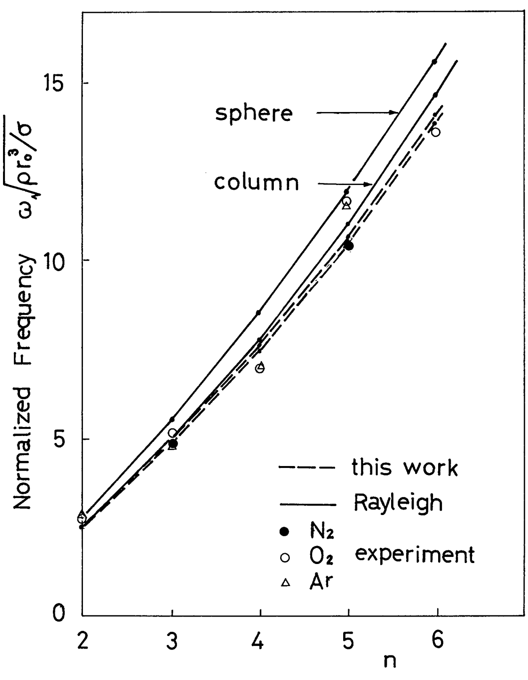

Fig. 3. Comparison of the experimental values with the present result and the Rayleigh's formulae. The width in the present theoretical values comes from their dependence on H (experimental values of H are used). Average radius r_0 in the plane view is used in normalizing the frequency.

our theoretical results depend on H, experimental values of H is used to obtain Ω, hence our results are presented with a small but finite width. The same values of $(\sigma/\rho)^{1/2}$ are assumed as in Fig. 2.

As for $n=2$, the best agreement is attained by Rayleigh's formula for sphere. This fact is well understood since this mode appeared in relatively small drops which, owing to strong effect of the surface tension, had an average shape of sphere rather than a flattened disk. Since the Rayleigh's result is reliable for a spherical drop, i.e. for $n=2$ vibration, this agreement is considered to support the assumptions on values of the surface tension coefficients, i.e. they are smaller than the standard ones by about 10%.

For $n\gg 3$, the Rayleigh's result for a sphere deviates from the experiment, and both the present theory and the Rayleigh's for column agree to it. Since the experimental values are rather scattered, it is difficult to decide which of these is better. However, it is at least certain that the present theory based on more realistic assumptions on the drop geometry attains a good agreement. If experimental data for larger n are obtained, the situation will become clearer, since the present result (eq. (27)) claims that $\Omega\propto n$ for $n\gg 1$ rather than $n^{3/2}$ as in the Rayleigh's formula.

From the results of the present theory the following conclusions are made. First, the assumptions made in §2 for a flattened drop are appropriate to analyse its dynamics, i.e. the concept of the shallow water with the modified boundary, etc. Therefore, the set of equations derived on these assumptions (given in §2) is expected to be powerful also for analysis of nonlinear dynamics.

Secondly, the values of $(\sigma/\rho)^{1/2}$ for liquefied gases evaporating rapidly in the room temperature are considerably smaller than the standard values, i.e. by about 10%. This conclusion, however, is still controversial, since the present theory is based on several assumptions, for example, the gas flow near the drop surface is neglected, and since the theoretical result, valid for $H\ll 1$, is applied to the region $H\sim 0.5$. Nevertheless, the present results strongly suggest us to have a new aspect for material constants in highly inequilibrium state. Note that the proposed values of $(\sigma/\rho)^{1/2}$ are near to those derived in the previous paper[8] by a much simpler theory. Generally speaking, frequency measurement of a flattened drop will be a good method to know the surface tension coefficient at overheated condition.

On the other hand, some of interesting phenomena observed in our experiment are not solved by the present theory, i.e. how the vibration begins, how the transition to another mode occurs as the drop radius is reduced, or why a distinction between even and odd modes exists. These problems will be attacked by the present authors by developing a nonlinear theory and will be presented in the later issues of this series of papers.

Acknowledgements

The present author would like to express their cordial thanks to Professor A. Arima for his interest in this work.

This work is partially supported by the Grant-in-Aid from the Ministry of Education.

Appendix

Two curvatures in eq. (7) are derived here up to the first order of perturbation. The first of them is straight forward, i.e.

$$\frac{1}{R_1}=\frac{2}{h}=\frac{2}{R_0H}\frac{1}{1+\tilde{h}}=\frac{1}{R_0H}(1-\tilde{h}). \quad (\mathrm{A}\cdot 1)$$

The second radius of curvature R_2 is already obtained by Rayleigh,[3] but his result does not contain an explicit expression. Therefore, its derivation is given here. Let a point on the contour in a plane view of a drop be expressed by $\boldsymbol{r}=R^*(\theta)\boldsymbol{i}_r$, where $\boldsymbol{i}_r$ is the radial unit vector, and a length along the countour be denoted by s. Then, the Frenet-Serret's formula leads to

$$\boldsymbol{t}=\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}=\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s}=(R^{*\prime}\boldsymbol{i}_r+R^*\boldsymbol{i}_\theta)\frac{\mathrm{d}\theta}{\mathrm{d}s}, \quad (\mathrm{A}\cdot 2)$$

$$\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s}=-\frac{1}{R_2}\boldsymbol{n}$$

$$=(R^{*''}-R^{*})\boldsymbol{i}_r+2R^{*'}\boldsymbol{i}_\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2+(R^{*'}\boldsymbol{i}_r+R^{*}\boldsymbol{i}_\theta)\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2}, \tag{A·3}$$

where $\boldsymbol{t}$ and $\boldsymbol{n}$ are unit vectors tangent and normal to the contour, respectively, $\boldsymbol{i}_\theta$ is the circumferential unit vector and the prime denotes the derivative. By the use of the formulae

$$\frac{\mathrm{d}\theta}{\mathrm{d}s}=(R^{*'2}+R^{*2})^{-1/2},$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2}=\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)\right)\frac{\mathrm{d}\theta}{\mathrm{d}s}=-\frac{R^{*''}R^{*'}+R^{*}R^{*'}}{(R^{*'2}+R^{*2})^2},$$

and by neglecting second order terms of $R^{*'}$ and $R^{*''}$, one has

$$\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s}=\frac{R^{*''}-R^{*}}{R^{*2}}\boldsymbol{i}_r+\frac{R^{*'}}{R^{*2}}\boldsymbol{i}_\theta,$$

hence

$$\frac{1}{R_2}=\left|\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s}\right|=\frac{1}{R^{*}}-\frac{R^{*''}}{R^{*2}}. \tag{A·4}$$

Substitutions of eqs. (3) and (13) into eq. (A·4) lead immediately to

$$\frac{1}{R_2}=\frac{1}{R_0(1+\beta H)}\left(1-\frac{\tilde{R}+\beta\tilde{h}}{1+\beta H}\right)-\frac{R_0(\tilde{R}''+\beta H\tilde{h}'')}{R_0^2(1+\beta H)^2}. \tag{A·5}$$

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