Self-induced Vibration of a Drop

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Self-induced vibration of drops of liquid nitrogen and oxygen placed on a plate with room temperature was observed, and some theoretical studies of this phenomenon are made. The drop became flat owing to the gravity, and its plane shape showed a certain kind of vibration, where its peripheral shape had a standing wave. Moreover, as the drop size decreased through evapolation, the number of waves around the periphery decreased after sudden transitions. Theoretical analysis is made to predict normal mode frequencies for wave numbers 2∼6, which agree well with experimental values. A mechanism of self-induction is proposed, which is based on an assumption on vapor flow around the drop.
Key words: Super-Heated Drop, Drop Vibration, Self-Induction, Normal Mode

1. Introduction

Let a drop be placed on a horizontal super-heated plate. If it is a water drop, the plate must be heated higher than about 200°C, while if it is a drop of liquid nitrogen (boiling temperature is −196°C) or oxygen (−173°C) the plate with room temperature is suitable. Then, the drop is deforms to a circular disk owing to the gravity and the surface tension. The vapor is continuously supplied from the lower surface of the drop, which flows outwards through a thin layer between the lower surface and the plate (the thickness of this layer is guessed to be of $O(0.1\text{mm})$ or less). The pressure in the layer owing to this vapor flow is high enough to levitate the drop. A static state of the levitating drop without vibration is possible theoretically, but actually its shape begins to vibrate in the horizontal direction.

Theoretical studies of drop vibration with small amplitude have been made mostly for that of a sphere and a liquid column (Rayleigh, 1879, 1902), and the formula for the frequency of vibration of a spherical drop is also given in the textbook of Landau and Lifshitz (1987). The axisymmetric and nonsymmetric vibrational modes of a spherical drop are expressed by the use of the polar coordinate system (r, θ, ϕ) and the spherical harmonic functions as follows:

$$r(\theta, t) = R + a(t)P_l(\cos\theta),$$
$$\text{for axisymmetric case,}$$

$$r(\theta, \phi, t) = R + a(t)Y_{lm}(\theta, \phi)$$
$$= R + a(t)P_{lm}(\cos\theta)\exp(im\phi),$$
$$\text{for nonsymmetric case, (1)}$$

where P_l, P_{lm} are the Legendre and the associated Legendre funcions, respectively. The function $a(t)$ is expressed as

$\exp(-i\omega t)$, where

$$\omega^2 = \frac{\gamma}{\rho R^3} \cdot l(l-1)(l+2), \qquad (2)$$

R is the drop radius, γ and ρ are the surface tension coefficient and the density, respectively.

However, the present problem is the vibration of a drop which is flattened by the gravity, whose mathematical expression is much more complicated than the spherical and the cylindrical cases. In the following sections experimental results by the present author and his colaborators are shown. Then, some of theoretical results are introduced.

2. Observation of Vibrations of Flattened Drops

Figure 1 shows several frames from high-speed movies of vibrations of drops of liquid nitrogen and oxygen, which are reproduced from Adachi and Takaki (1984). In this experiment the drop was placed on a slightly concave lens, so that the drop did not flee away from the scope of the camera. In each case in this figure several shots of vibration during the half period are shown. There had been no other report on this phenomenon, except that by Holter and Glasscock (1952), who observed similar vibrations of liquid. It is noted here that the movies of this vibration were taken by stuffs of Laboratory of T. Uemura at the Institute of Industrial Science, The University of Tokyo, and these photos were first shown in a Japanese magazine for general people, "Suri Kagaku (Mathematical Sciences)", by Arima and Adachi (1967). The motivation of this study was a wish of A. Arima (a nuclear physicist) to visualize vibration of atomic nuleus, and K. Adachi (an experimental physicist) proposed a method to use liquefied gas.

In this experiment the period of vibration T changed within 30–50 ms, the average radius r_0 was 2 or 3 mm and the amplitude of vibration was nearly constant of $O(1\text{mm})$ in any run of experiment. The mode of vibration is specified by the number n of waves around the periphery. What is remarkable in this vibration is the following fact. As a drop
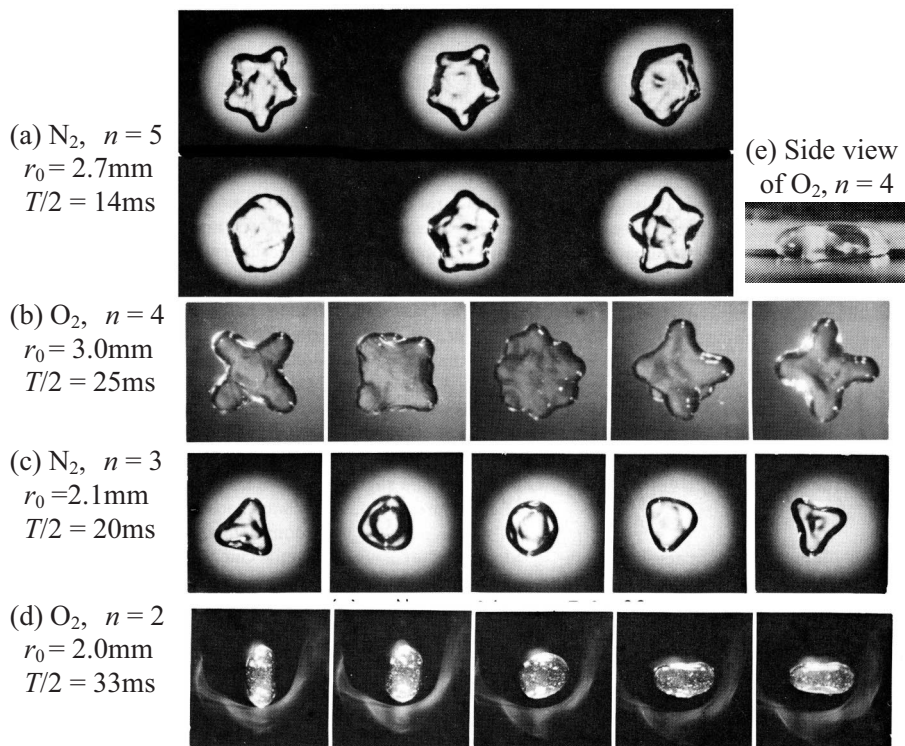
Fig. 1.   Various vibration modes of superheated drops of liquid nitrogen and liquid oxygen placed on a plate with room temperature. Instantaneous shapes during a half period of vibration are shown. The mean radii of drops r_0 and the half period $T/2$ are shown at the left. The drop of liquid nitrogen showed the modes with $n = 5, 3$ and 2 (the mode with $n = 4$ did not appear), while that of liquid oxygen showed all modes with $2 \leq n \leq 6$. (e) Side view of the flattened oxygen drop, whose thickness was $O(2\text{mm})$ (reproduced from Adachi and Takaki (1984)).
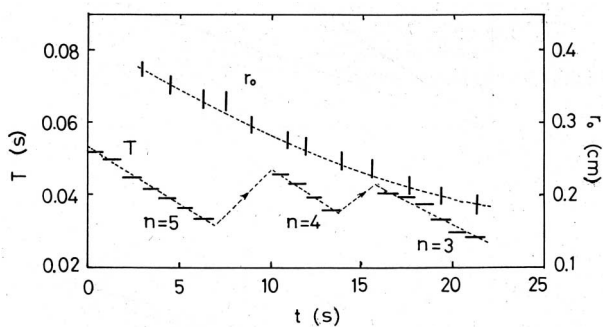

Fig. 2.   Transition of vibration mode of liquid nitrogen with changes of vibration period T and average drop radius r_0 (reproduced from Adachi and Takaki (1984)).
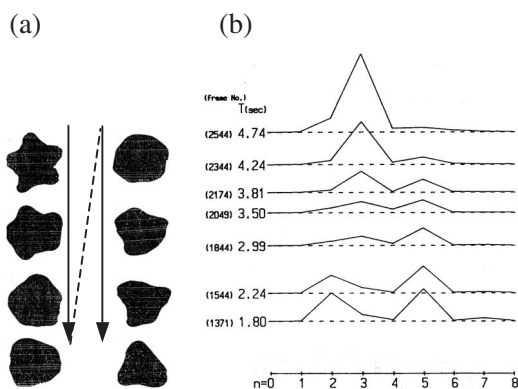

Fig. 3.   (a) Instantaneous drop shapes of liquid nitrogen during a transition from mode $n = 5$ to 3, and (b) the change of spectrum of the peripheral shape (from below to above) (reproduced from Takaki *et al.* (1989)).

became smaller through evaporation, it stopped to vibrate periodically and made transition to a new mode with smaller value of n. For a few seconds during this transition the drop showed complicated irregular motion, then it began a new regular vibration. This process is shown graphically in Figs. 2 and 3.

Figure 2 shows a process of mode transition for a drop of liquid oxygen along with the decrease of the mean radius r_0 and the change of vibration period T. During the intervals with constant values of n the drop vibrated regularly with slow degrease of radius and vibration period. On the other hand, during the tarnsitions to smaller values of n the drop showed irregular motions with asymmetric shapes, an ex-

ample of which is shown in Fig. 3(a). Takaki *et al.* (1989) obtained the change of spectrum of peripheral shape of drop expressed as a function $r = f(\theta)$, as shown in Fig. 3(b). In the first stage (the bottom) the spectrum has two peaks at $n = 2$ and 5; i.e. the drop was vibrating as a superposition of two modes $n = 5$ and $n = 2$. Through an interaction between these modes the drop acquired the $n = 3$ mode as shown in the spectrum of the last stage (the top).

The mechanism of mode transition is not yet clarified enough, and following problems are left unsolved. First, why does the drop not continue to keep a particular mode,
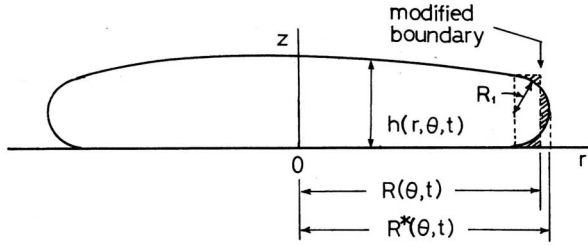
Fig. 4. Side view of the drop and definition of the modified boundary $R(\theta, t)$ based on the real radius $R^*(\theta, t)$ in the plane view.

i.e. why does it make a transition? This problem has not been treated until now, but a qualitative speculation might be possible. Let us assume that the drop was keeping a constant wavelength λ through a certain mechanism, and a wave number $n = 2\pi r_0/\lambda$ was chosen, hence the wave number n decreased as the radius R decreased.

Secondly, why the nitrogen drop with vibration mode $n = 5$ skipped the mode $n = 4$ and made a transition to mode $n = 3$, while the oxygen drop took every mode? In macroscopic dynamics such as the drop vibration the difference of materials should affect the dynamics only through material constants such as the density ρ and the surface tension coefficient γ. However, effects of these constants are nearly the same for nitrogen and oxygen. One of the collaborators of the present author made an experiment to observe the vibrations of liquid nitrogen placed in various temperature environments between $-100°C$ and $200°C$ (Yoshiyasu *et al.*, 1996). He tried more than 50 runs and found that for the ambient temperature higher than $150°C$ the drop showed various vibration modes randomly (not dependent on the drop size) and that the drop sometimes made vibrations with $n = 4$ mode. Until now, this problem is not solved and remains as a mystery!

3. Theoretical Analysis of the Drop Vibration

A linear analysis of normal mode vibration is made by Takaki and Adachi (1985) by simplifying the vertical cross section of the drop. In the analysis the polar coordinate (r, θ) is used in the horizontal plane, and the following three assumptions are introduced:

1. The radius of the drop in the plane view is much larger than the drop thickness, so that the velocity $\boldsymbol{u}(r, \theta) = (u_r, u_\theta)$ inside the drop has nearly horizontal direction, and the thickness distribution $h(r, \theta, t)$ is a slowly varying function of the coordinates.

2. The viscosity of the liquid is neglected and the velocity of liquid is expressed by a flow potential function $\phi(r, \theta, t)$, so that $u_r = \partial\phi/\partial r$, $u_\theta = \partial\phi/r\partial\theta$.

3. An effective periphery of the drop in the plane view, called modified boundary and denoted by $r = R(\theta, t)$, is introduced, which is related to the real periphery $r = R^*(\theta, t)$ by the following equation:

$$R^*(\theta, t) = R(\theta, t) + \beta h(R, \theta, t),$$
$$\text{where } \beta = 1/2 - \pi/8 = 0.107. \quad (3)$$

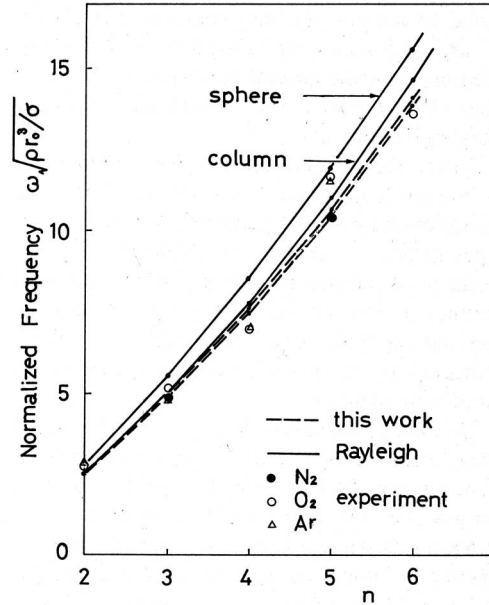The term $\beta h(R, \theta, t)$ indicates the shift of the periphery, as

Fig. 5. Theoretical and experimental results of normalized vibration frequencies of a flattened drop (denoted by Ω in the main text). The dashed curve shows the present result (Eq. (11)), and two solid curves are by Rayleigh (1879, 1902) (reproduced from the paper by Takaki and Adachi (1985)).

shown in Fig. 4, so that the half circle at the periphery is replaced by a rectangle with the same area.

At the modified boundary the following kinematical condition should be satisfied:

$$\frac{\partial\phi}{\partial r} = \frac{\partial R}{\partial t}, \text{ at } r = R. \quad (4)$$

i.e. the outward fluid velocity at the periphery must match the motion of the boundary. On the other hand, the horizontal force P owing to the surface tension at the periphery is expressed as

$$P = h\gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \quad (5)$$

where $R_1 = h(R(\theta, t), \theta, t)/2$ and R_2 is determined from the real periphery $r = R^*(\theta, t)$. Here, the real radius is expressed as a superposition of the mean value and perturbation, i.e. $R_2 = R^*(\theta, t) = R_0(1 + \beta H) + \tilde{R}(\theta, t) + \beta H\tilde{h}(R_0, \theta, t)$, and its inverse is expanded to the first order as follows:

$$\frac{1}{R_2} = \frac{1}{R_0(1 + \beta H)} - \frac{1}{R_0(1 + \beta H)^2}$$
$$\cdot\left(\tilde{R} + \beta H\tilde{h} + \frac{\partial^2}{\partial\theta^2}(\tilde{R} + \beta H\tilde{h})\right). \quad (6)$$

In the following the length and the time are normalized by the average radius of the modified boundary R_0 and $T_0 = (\rho R_0^3/\gamma)^{1/2}$, respectively, where the same notations r and t are used for normalized variables. Then, the fluid motion within the drop is governed by the continuity equation and the Euler equation (the Navier-Stokes equation without the viscosity term) for the thickness $h(r, \theta, t)$ and the velocity
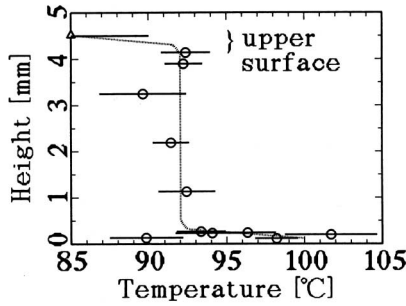
Fig. 6. Distribution of temperature within a water drop placed on a hot plate with temperature 320°C. The height was measured from the surface of the hot plate (reproduced from Tokugawa and Takaki (1994)).
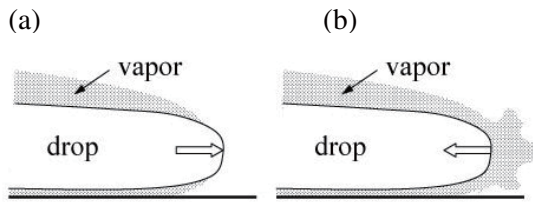
(a)   (b)

Fig. 7. Behavior of the vapor around the drop. When the periphery of the drop moves outwards (a), the vapor at the periphery was wiped away and the surface is directly contact to the hot environment, while in the opposite motion (b), it was covered by the vapor and insulated from the environment.

vector $\boldsymbol{u}(r, \theta, t) = (u_r, u_\theta)$, as follows:

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial (rhu_r)}{\partial r} + \frac{1}{r}\frac{\partial (hu_\theta)}{\partial \theta} = 0, \tag{7}$$

$$\frac{\partial (\rho h \boldsymbol{u})}{\partial t} + \frac{\partial (u_r \rho h \boldsymbol{u})}{\partial r} + \frac{1}{r}\frac{\partial (u_\theta \rho h \boldsymbol{u})}{\partial \theta} = -\mathrm{grad}\,P, \tag{8}$$

where $P = \rho g h^2/2$ is a vertical integration of the pressure owing to the gravity (the effect of surface tension on the upper surface is neglected).

In order to obtain the frequency of normal mode vibration with mode n, the flow potential, the thickness function and the position of modified boundary are expressed as

$$(\phi, h, R) = (0, h_0, R_0) + \left(\frac{R_0}{T_0}\tilde{\phi}(r), h_0\tilde{h}(r), iR_0\tilde{R}\right)$$
$$\cdot \exp(in\theta + i\Omega t), \tag{9}$$

where h_0 is an average thickness of the liquid. Solution of the Eqs. (7) and (8) is obtained as follows:

$$\tilde{\phi} = AJ_n(wr), \tag{10}$$

where $J_n(wr)$ is the Bessel function, A is an arbitrary constant, $w^2 = \Omega^2/GH$, $H = h_0/R_0$ and $G = \rho g R_0^2/\gamma$. By substituting this solution into the boundary conditions (4)~(5) and expanding the Bessel function in terms of wr, we obtain an eigenvalue of Ω^2, as follows:

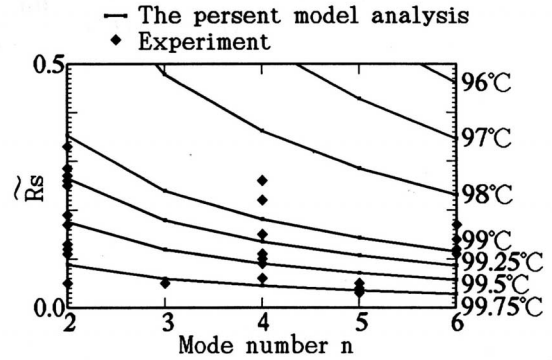$$\Omega^2 = n(n^2 - 1) \Big/ \left\{(1 + 2\beta H) + \frac{n-3}{2GH}\right\}. \tag{11}$$

Fig. 8. Predicted amplitudes $\tilde{R}_S$ of vibration for various surface temperatures in the period shown in Fig. 7(b) and the experimental values of amplitude (reproduced from Tokugawa and Takaki (1994)).

This result is shown in Fig. 5 by dashed line along with the measurement by Adachi and Takaki (1984) and the results of Rayleigh's analyses (1879, 1902) for the vibration frequencies of a spherical drop and a circular column. The theoretical result by Rayleigh for a sphere is given as Eq. (2), while that for a circular column is concerned to a nearly two-dimensional motion perpendicular to the cylinder axis and is given by

$$\omega^2 = \frac{\gamma}{\rho R_0^3} \cdot n(n^2 - 1), \tag{12}$$

where R_0 is the cylinder radius.

The present result agrees with experiment better than those by Rayleigh. This difference is considered to come from the difference of drop shapes, i.e. Rayleigh's analyses are concerned to a sphere and a circular column while the experiments were made for drops of circular disc. However, the Rayleigh's result for the circular column agrees better than that for the sphere. In general, theoretical results might be strongly governed by the number of dimensions of objects.

4. Mechanism of Excitation of Vibration

Until now the normal mode of vibration of a drop has been discussed based on a relatively simple linear analysis, and the mechanism of excitation of vibration was not treated. In order to attack this problem, a behavior of the vapor around a vibrating drop should be examined. An experimental and theoretical studies of were made by Tokugawa and Takaki (1994) with special interest in the behavior of this vapor, as is introduced briefly below.

First, they measured the temperature distribution in a water drop on a hot plate with temperature 320°C by the use of a thermocouple, the result of which is shown in Fig. 6. The drop did not begin to vibrate due to the presence of the temperature probe. The temperature was nearly at the boiling point on the lower surface, while that in the other parts was within 90~95°C. It suggests that there was a vapor layer covering the drop, although its thickness was unknown.

According to this result it is assumed that this vapor plays a role of insulator which separates the drop from the sur-

rounding hot envirommrnt. Now, when a part of periphery went outwards, the vapor at the periphery would have been wiped away and the surface was directly contact to the air and also received a thermal radiation from the hot plate, as illustrated in Fig. 7(a), and the surface temperatue would have been hot. When the periphery went inwards (Fig. 7(b)), it was covered by the vapor and the liquid surface was insulated, hence its temperature would have been about 92°C (see Fig. 6). Since the surface tension coefficient is smaller for higher temperature, the periphery receives a varying surface tension during one period of vibration, so that the surface tension coefficient is assumed to depend on the velocity at the periphery as follows:

$$\gamma = \begin{cases} \gamma_0, & \text{for } d\tilde{R}/dt \geq 0, \text{ (a)} \\ \gamma_0(1+q), & \text{for } d\tilde{R}/dt < 0, \text{ (b)} \end{cases} \qquad (13)$$

where q is a positive constant depending on the surface temperature. This effect is considered to be a major nonlinear effect.

In addition, a viscous effect of the horizontal flow within the vapor layer below the drop is considered here, which acted as a damping effect balancing with the amplifying effect of the nonlinearity. On the other hand, this viscous stress is estimated from the balance of the gravity force due to the drop and the pressure at every point in the layer, where the horizontal gradient of this pressure gives the flow in the vapor layer hence the viscous damping effect.

The mathematical expression of this analysis is much complicated and not shown here. Result of the nonlinear analysis is shown graphically in Fig. 8. Since the surface temperature at the periphery in the period shown in Fig. 7(b) is unknown, its value is assumed within the interval (95°C, 99.75°C), and the predicted amplitudes $\tilde{R}_S$ are compared with the experimental ones. Although the latter is rather scattered, the case with 99.25°C for the surface temperature fits best to the experiment, which correspond to $q = 0.0024$. It is remarkable that this very small fraction of surface tension variation is enough to produce a steady vibration with large amplitude.

By the way, it is noted here that the self-excited vibration of a drop is observed when it is placed on a plate oscillating vertically, and a theoretical analysis based on Mathieu equation is made (Yoshiyasu *et al.*, 1996). Its plane view is quite similar to those shown in Fig. 1.

5. Concluding Remarks

The drop vibration introduced here is an example of the dssipative structures in a broad sense, because the vibration is maintained by the heat transfer from the hot plate to the surrounding air. However, the mechanism of the maintenance of this dynamical sturucture is much more complicated than in the cases of simple heat conduction and crystal growth. It is considered to come from the fact that the temperature difference between the higher and the lower sides is large, i.e. the difference has nearly the same oder of magnitude as the lower temperature. In many of such cases the system is considered to allow more than one dynamical states, and the governing mechanism plays a role to choose one of them, just as the vibrating drop discussed above choses one preferrable mode with a particular value of n. However, it depends on the present author's speculation, and we have no general knowledge on the cases with highly inequilibrium condition. In order to have the general knowledge more number of phenomena with this kind of condition must be investigated.

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