A chain that accelerates, rather than slows, due to collisions: how compression can cause tension

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When two objects collide their velocities change in response to the compressive (pushing) force between them. The difference in (normal) velocities between the objects is thus eliminated or reversed. However, for non-rigid objects collisions are more subtle. Surprisingly, when a long chain moving lengthwise collides with, say, a wall or floor, the chain can be pulled into the wall (instead of pushed away) with the approach velocities between the wall and chain increasing in time (rather than not changing or decreasing). Why? The incremental bits of mass that are colliding are slowed by the wall. But they can also be slowed by the remaining chain, thus speeding the remaining chain. The extent to which the impulse which slows the colliding bits comes from the wall or from the remaining chain determines the acceleration of the remaining chain. We show theoretical limits on how much a chain can be pulled into something with which it collides, some chain link designs that lead to these limits, and experimental results which show the sucking of one of these designs into a wall.

I. INTRODUCTION

A common textbook 'variable mass' problem is:

> **Classic problem:** *A vertical chain hangs just above a table. It is dropped. What is the force on the table?*

This and related problems are meant to show application of momentum balance to variable-mass systems.[1–5] The intended calculation has the chain falling downwards with acceleration g. The force on the table is the weight of the chain on the table plus the momentum flux of the chain coming to rest as it collides. As the chain falls, the flux grows as the chain speed increases, and the weight of accumulating chain also increases. Thus the desired answer:

> **Classic Answer:** *While the chain falls the force on the table is three times the weight of the fallen portion.*[1]

Experiments with chain generally show reasonable agreement with this theory.[6–8] Despite the simplicity of the theory, and the confirmation by experiments, there are subtleties. The theory has hidden assumptions which we will discuss throughout the rest of the paper: the upper chain can actually fall with acceleration greater than g. We first review our history with these experiments, and also the related literature. We then describe some theoretically bounding cases. Finally, we present a new experiment which shows a chain being pulled into the surface with which it collides.

A. The persistent student

We were first tripped up by a chain problem when solving one at the blackboard for a sophomore engineering dynamics class at Cornell in 1984. We are generally dogmatic about basing mechanics reasoning on free body diagrams; we insist that any use of momentum balance must be based on a (real or imagined) picture of the system and all the external forces acting on it. We had drawn a free body diagram of the colliding link and shown the collision force of the ground on that link. We were caught out by a student who said, roughly:

> **Student:** *You told us to draw a force at any point where you have cut your system free from its environment. You cut the last link free from the chain above it, why don't you show a force there? Why doesn't the last link pull on the chain above when it hits the table?*

The question was annoying. Obviously we can use the

> **Key classic assumption:** *The last link is pushed up from the table and is thus released from the falling links above. There is no interaction force (or impulse) between the colliding link and the chain above.*

Then to our delight, and hopefully yours, we realized that the assumption that the last link breaks loose from the chain above is just that, an assumption. Certainly it is generally a reasonable approximation. However, if there was a force between the colliding link and the chain above, then the chain above would fall with acceleration greater than g. For such a chain all the classic calculations would be wrong. We designed, but didn't build, some chains where this key classic assumption was violated. We called the authors of the textbook in use at the time,[5] gave a seminar on the theory, mentioned it as a puzzle to various people who like mechanics puzzles, and let the problem sit.

As discussed in more detail below, in the intervening years various others have made related discoveries for related chain problems and also done more careful experiments revealing related discrepancies. Our purpose here is to fill in a few features of the problem not yet filled in by the literature between 1984 and the present, and to describe our confirming experiment which differs in some details from others published so far.

First, we review the classic and more modern falling-chain literature.

II. THE CLASSIC FALLING-CHAINS

Various chains and ropes are used for these 'variable-mass' problems.[1–5,9–13] In all variants there is a moving part of chain and stationary part. The moving part falls 'freely' or is moved with prescribed force or velocity. The stationary part is contacting or connected to an immovable wall or floor. The three standard geometries we will call the 'bottom-pile' (our main concern here), the 'top pile' and the 'U-chain' (see Fig. 1).

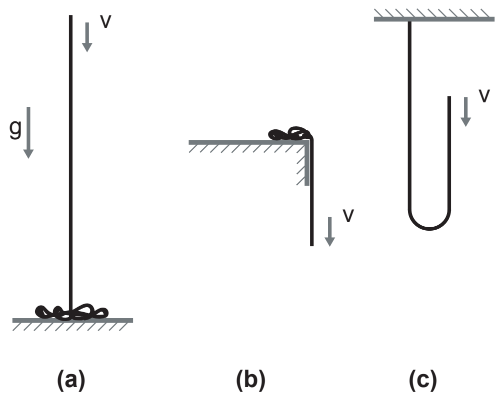

FIG. 1: The three standard chain configurations: (a) 'Bottom-pile' chain. (b) 'Top-pile' chain. (c) 'U-chain'. In all three cases there are two chain segments, one stationary and one moving. Depending on the direction of motion (sign of v), the links transfer from moving to still or vice versa when leaving one segment and joining the other.

Bottom-pile chain. A pile of chain rests on the ground and a vertical segment of that chain is either lowered on to the pile or lifted from it. Either the force or velocity is specified.

Top-pile chain. A pile of chain is on a table and a vertical segment of that chain falls through a hole or off the edge, as pulled by gravity.

U-chain. In the 'U-chain', also sometimes called the 'folded chain', one end of the chain is held fixed and a stationary vertical segment hangs from it. The other end is lifted up to form a U shaped fold at the bottom. As this end is let go or moved with prescribed force or velocity, mass passes through the fold from the moving to the stationary segment or vice versa, depending on whether the chain is lowered or lifted. Gravity may or may not be present.

In all cases one may choose to model the chain either as a continuum or with discrete links. The calculations are easier in the continuum case, but the concepts are more clear with discrete chain links. The following core concept is not in doubt:

> **Key fact:** *As a link transitions from moving to stationary, or stationary to moving, an impulse (or a large force acting for short time) must be applied to it.*

In all classic treatments, the collisions are taken to be plastic (or inelastic, the colliding link matches the velocity of the segment it joins). The pile shape and details of the falling geometry are usually ignored. In most classic treatments the authors take it as self evident where the collision impulse comes from: it comes from the segment the new link is joining. In more detail, the classically assumed sources of impulse on the transitioning link are as follows.

Falling bottom-pile chain: The ground provides an impulse to the impacting link and arrests its motion.[1–8,11,14] In this case the tension in the falling portion is just from weight, or zero if the chain is falling freely.

Lifted bottom-pile chain: Each link is accelerated into motion from an impulse caused by the chain which lifts it.[4,5,9,10,13]

Falling top-pile chain: The already falling segment of chain causes an impulse on the next link to join, accelerating it into the falling motion.[2,4,5,9,13,15,16]

Lowered U-chain: The lowered link's motion is arrested by an impulse from the hanging segment.[2,3,5,7,12,14,17–24]

Raised U-chain: The link changing from hanging to rising is accelerated by an impulse from the rising segment of the chain.[5,12]

A typical and thorough treatment of the classic approaches (above) of various discrete chain problems is in the recent edition of Meriam and Kraige.[5]

III. QUESTIONING THE ASSUMPTIONS

Some aspects of the classic approaches above have been questioned, as described now.

The U-chain. That the U-chain might be considered non-dissipative was already understood, implicitly, by Routh in 1898.[25] Although not in the context of collisions per se, Routh presents the beautiful result that a continuous rope or chain of any shape has a constant shape dynamic solution wherein it moves with constant speed tangent to the shape. This solution is invoked, again implicitly, in the solution for cracked whips and fishing line.[26,27] For the U-chain, the whip and the fishing line the propagating shape (U or loop), as viewed in the reference frame of the traveling loop, is the Routh constant-shape solution. The Routh solution has a tension $T = \rho v^2$ where ρ is the mass per unit length. This is the tension which pulls on the moving portion of the chain, thus accelerating it.

Wong's thorough review of U-chain problems notes that Hamel was the first to explicitly invoke energy conservation for U-chains.[24,28]

The first paper that seems to explicitly ponder whether, or how much, a real physical chain should be considered dissipative or not, is a brief mention of the U-chain by Satterly in 1951.[7] On the one hand he doubts the classic approach (far above) 'the assumption of an acceleration g for the falling chain ... might be too rash ... ' On the other extreme, he also says 'it is a bit dubious to employ the conservation of energy in impulse problems.'

For the U-chain various authors have considered the energy conserving model and found that it matches reasonably well with experiments.[5,17,20,22,24] The U-chain is conceptually the simplest of the three chains in Fig. 1 because a continuum model of the transition region can be accurate (if the U is wide) and can be directly analyzed. When folded tight, however, the continuum approximation is questionable.[20] In this tight-fold case, even in a discrete chain there are no made or broken contacts and thus no explicit collisions. Discrete chain simulations and experiments show that there is little residual vibration in the nominally stationary or moving portions of the chain. Thus macro-scopic energy conservation is found to nearly hold.

Top pile chain. Wong *et. al.* recently reviewed the top-pile chain.[16] In their experiment the pile was aligned parallel to and at the edge of table. They found reasonable agreement with energy conservation. Incorrectly, we feel, Wong *et. al.* deduce energy conservation by appeal to Lagrange equations. By assuming that Lagrange equations apply they have already implicitly assumed energy conservation. (It seems that by similar reasoning they could conclude that a block sliding on a plane necessarily has zero friction or that all colliding bodies have a coefficient of restitution = 1.)

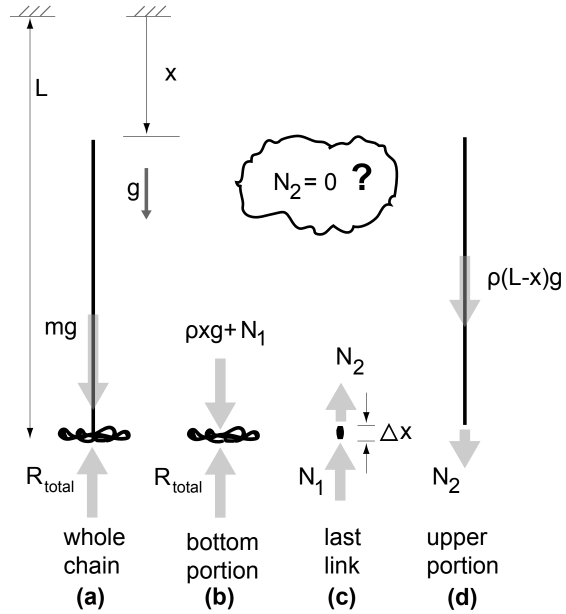

FIG. 2: Free Body Diagrams for portions of a lowered bottom-pile chain. (a) Chain as a whole with weight, mg and reaction from table, R_{total}. (b) Part of the chain (length x) resting as a pile on table. (c) Last link (or the jump region), where transition from moving to stationary occurs. (d) Segment of the chain in the air. The central question here concerns N_2, the tension induced by the collision between the colliding link and the chain above. Can one can fairly assume that $N_2 = 0$ and if not, how big might it be?

The bottom pile chain. In their more recent editions (triggered by our phone call in 1984?) Meriam and Kraige consider falling of a smooth rope for which they use energy conservation. For the lifted bottom-pile rope (sample prob. 4/10), they propose a massless and frictionless feeder as a model of the rope being lifted smoothly, and without dissipation, from a well arranged pile.[5]

Most investigations of the bottom pile chain assume the upper chain falls with acceleration g, as per the classic assumptions above.[6–8] However, recent and more careful experiments by Hamm and Geminard have revealed that the bottom-pile ball-chain falls a bit faster than g experimentally.[29] Using a continuum model with finite curvature in the contacting region, they derive the presence of some tension on the vertical segment of the chain (their mechanics implicitly finesses use of an un-mentioned bending stiffness and a consequent superposition of the Routh solution above with an un-mentioned 'elastica' solution).

IV. ANALYSIS OF A BOTTOM-PILE CHAIN

We attempt to clarify the issues raised above. As an example we focus on the bottom-pile chain with one dimensional motion. The total length is L, the mass m and the density $\rho = m/L$. The length of chain accumulated on the table is x with mass $m_x = \rho x$). The length of the transition region (assumed small, this represents, say, the last link) is Δx with mass $\Delta m = \rho \Delta x$). Lets take N_1 and N_2 (Fig. 2) to be the average forces acting on the transition region over the transition time Δt. N_1 comes from the table (and the pile accumulated on the table). N_2 is from the chain above. The main points, expanded below, are that

- Linear momentum balance is not enough to solve such problems. And
- Energy balance is not enough of a supplement to solve them either.

Rather, the solution depends on the collision mechanics. These need to be worked out in detail or described with an appropriate collisional constitutive relation (continuum jump condition).

A. Proper free body diagrams: is $N_2 = 0$?

It seems self-evident that the collision of the bottom link with the table separates that link from the link above, so N_2 should be zero. This is the intuition behind what is called the 'complementarity' conditions in collisional mechanics. And these conditions are not just taken as reasonable by most authors, but as accurate. But this is a mistake, there are clear counter-examples.[30] In the case of the chain, what if the collision of the bottom link excited vibrations in that link and those vibrations caused, before separation, a momentary increase in the contact force between that link and the link above it? This would transmit a force N_2.

One possible rebuttal is that the chain links are modeled as rigid, so there is no place for considerations of such vibrations. However, if one is modeling the chain links as generally rigid, still there is no fundamental reason to exclude a force or impulse N_2; one can't apply the assumption of non-deformation *during* the collision,[31] and it is during the collision that N_2 is said to vanish. Or not.

In addressing such chain problems one may not want to get involved with the details of the mechanics of the links. Who is to say what kind of mechanics might apply during the collision of some chain links of unspecified design? Thus, again, we are forced to consider that N_2 might be present. Between 1984 and the present, various authors have noted the possibility that $N_2 \neq 0$ (or some equivalent expression) , especially in regard to the U-chain.[18,21,23,29,32,33] Any solution which invokes energy conservation is implicitly assuming that $N_2 > 0$. The theory and experiments presented below support the results in these papers, in that we agree that it is possible that $N_2 > 0$.

B. Momentum balance

The linear momentum balance (LMB) for the transition region, the last link, is

$$\begin{aligned} \text{Impulse} &= \Delta\text{Momentum} \\ (N_2 + N_1)\Delta t &= (\text{Mass}) \cdot \Delta v \\ (N_2 + N_1)\Delta t &= (\rho \dot{x} \cdot \Delta t)\, \dot{x} \\ N_2 + N_1 &= \rho \dot{x}^2. \end{aligned} \tag{1}$$

This is the so called Rankine-Hugoniot jump condition for shock propagation.[18,21] It says that the net force matches the momentum flux. Here, and below, we use the language of links but write continuum equations.

Linear momentum balance for the chain portion above the colliding link gives:

$$\sum F = ma \qquad \text{(down is positive)}$$
$$\underbrace{\rho(L-x)g}_{\text{weight}} + N_2 = \underbrace{\rho(L-x)}_{\text{mass}} \underbrace{\ddot{x}}_{\text{acceleration}} . \tag{2}$$

If N_2 is zero and the top part of the chain falls freely then

$$\ddot{x} = g. \tag{3}$$

In that case (3) can be integrated with zero initial velocity (because the chain was released from rest) and plugged into (1) to give $N_1(t)$. One can calculate the total table reaction (which according to Fig.2b, is N_1 plus the weight of the chain piled on the table) as:

$$\begin{aligned} R_{total} &= N_1 + \rho x g = 3x\rho g \\ &= 3 \cdot (\text{weight of chain on the table}). \end{aligned} \tag{4}$$

This is the classic solution quoted at the start of the paper. From (3), the length of chain in the pile x, increases parabolically in time from zero to L, and correspondingly the R_{total} also goes from 0 to $3mg$.

However, if we don't assume that $N_2 = 0$, combining (1) and (2) we get:

$$\ddot{x} = g + \left(\frac{N_2}{N_1 + N_2}\right)\frac{\dot{x}^2}{L-x}. \tag{5}$$

So the problem is actually not determinate, at least not without adding extra assumptions that determine N_2.

C. Assume positive mechanical energy dissipation

The table reaction force R_{total} doesn't do any work on the chain because the material point-of-application is at rest on the table surface. Assuming no heat transfer the system is adiabatic and the first law of thermodynamics applied to the whole chain, for a small interval of time from t to $t + dt$, gives:

$$d\left(U + \rho g (L-x)^2/2 + \rho(L-x)\dot{x}^2/2)\right) = 0, \tag{6}$$

where U is the internal energy of the chain and next two terms are gravitational-potential and kinetic energies respectively. Plugging (1) and (2) into (6) one gets:

$$dU = \frac{1}{2} dx (N_1 - N_2). \tag{7}$$

A thermodynamic argument for the equivalence of macroscopic mechanical energy dissipation and positive entropy production is given in O'Reilly and Varadi.[33] Alternatively we can take it as a postulate for this system that the macroscopic mechanical energy is non-increasing so $dU \geq 0$ and thus

$$N_1 \geq N_2. \tag{8}$$

When there is no dissipation, and mechanical energy is conserved, then $N_1 = N_2$ ($\dot{U} = 0$). In this non-dissipative limit N_2 transmits all of the kinetic energy of the last link to the upper portion of the chain. That energy conservation implies $N_1 = N_2$ was noted before for the U-chain.[17,33]

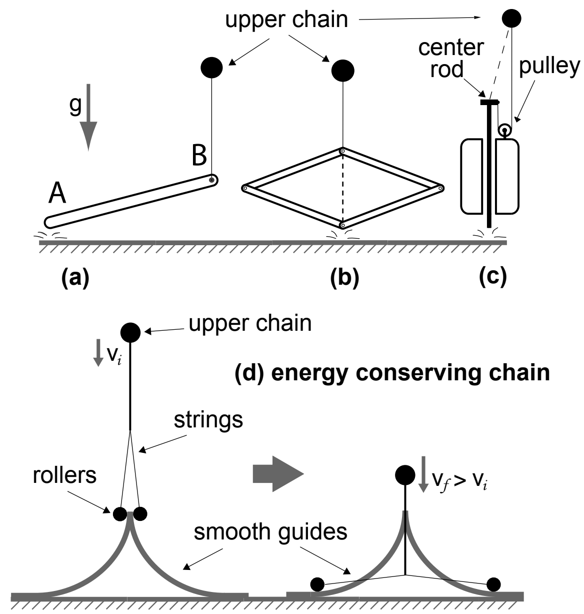

FIG. 3: Four different link designs that give $N_2 > 0$. **(a)** A slightly tilted rod hits the ground at a small angle. The other end would speed up if not connected to the upper chain. The mechanics of this design are explained with reference to Fig. 4. Our experiments are based on this design. **(b)** Each link is a 4-bar deforming diamond. When the bottom corner hits, the top corner would accelerate but for its connection to the upper chain. The connecting string (shown dashed) holds the diamond when hanging. **(c)** Each link has a mass, a massless rod and a pulley. When the rod hits the ground it stops. Assuming no jump in the upper-chain speed, the pulley kinematics dictates a halving of the mass speed. Half of the impulse needed for this comes from the string connected to the upper link, thus pulling on the upper chain. Later the mass hits the ground. The dashed cord holds the chain together when hanging. **(d)** An energy conserving chain. Each link has two rollers. As they fall down they frictionlessly move down the guides and spread out, coming to rest smoothly. Their energies are fully transferred to the upper chain.

V. SEARCH FOR CASES WHERE $N_2 \neq 0$

Can a discrete bottom-pile chain be made where N_2 is not zero? Consider a collapsing building, triggered by a ground floor explosion for example, as the lowest floor hits the ground the ground reaction force resists the downward motion of the floors falling above. For a falling building the slowing of the collapse of the upper part is due to $N_2 \neq 0$, in fact with $N_2 < 0$ as per the sign convention in Fig.2. Of course buildings are different than chains in that they can support compression, but in terms of our basic energy, entropy and dissipation arguments, the collapsing building is like a falling chain. There are no fundamental limits on how negative N_2 can be. The limit $-N_2 \to \infty$ corresponds, say, to the whole upper portion having an instantaneous decrease in velocity (e.g., coming to a stop like a rigid object).

The more counterintuitive regime is where $N_2 > 0$ (but restricted to $N_2 < N_1$) in which the colliding link is pulling down the upper portion. The next section discusses link designs for which $N_2 > 0$.

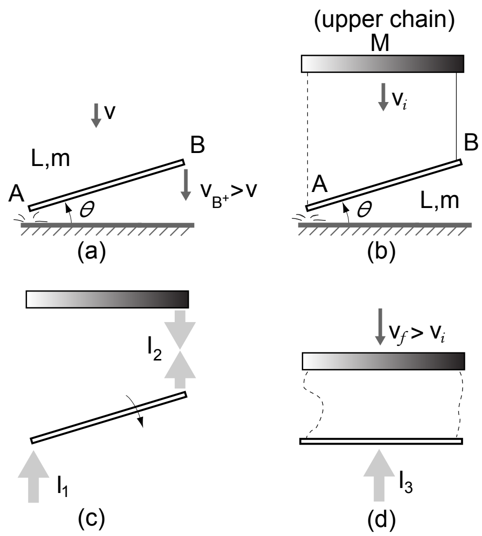

FIG. 4: Mechanics of the 'sucking' chain from Fig. 3a. (a) End B of a rod falling onto a surface speeds up when the other end A collides with surface. (b) In a simple model of a chain composed of such rods as links, when the end A collides the end B pulls down the upper chain via the connecting string. The string connecting the end A to upper chain slacks. (c) Free body diagram showing impulses at first collision when the link just collides. (d) Impulses when the link has its second collision with ground and finally comes to rest. (Assuming sticking-collisions)

A. Designs for a 'sucking' chain, with $N_2 > 0$

Fig. 3 shows four link designs where the colliding link pulls down on the ones above ($N_2 > 0$). Our experiments used the first of these.

Consider a uniform rod of mass m and length L inclined at a small angle θ moving vertically downward with velocity v towards a rigid surface (Fig. 4a). First consider this 'link' as not connected to an upper chain. When the end A hits the surface the other end B speeds up as required by angular momentum balance about point A. Assuming a sticking collision (and small angles), the downwards velocity of B increases to

$$v_B^+ = 3v/2 > v. \tag{9}$$

This means the second collision when, B hits the ground, is faster than was the collision at A, and a clattering sequence will ensue. This is a consideration for, say, design of impact resistant cell phones.[34] Now consider this link connected to an upper chain made of many such links. Fig. 4b. shows such a chain. The upper part of chain is lumped into a mass M which is large compared to the link mass m. As the end A undergoes a sticking collision (assumption), the ground provides a normal impulse I_1 and end B pulls on the upper chain with an impulse I_2 (Fig. 4c). Finally the last link hits the ground again and looses all its momentum by an impulse I_3 (Fig. 4d). Using linear and angular momentum balance, along with post-collisional equality between velocity of end B and the mass M, these impulses are, assuming $\theta \ll 1$:

$$\begin{aligned} I_1 &= mv/3 && \text{collision of A} \\ I_2 &= mv/6 && \text{collision of B} \\ I_3 &= mv/2 && \text{final collision with the ground.} \end{aligned} \tag{10}$$

I_2 is of interest here as the impulse associated with our N_2 ($I_2 = \int N_2 dt$).

In obedience to the quasi-thermodynamic restriction (8), the total impulse from the ground ($I_1 + I_3$) to bring the link to rest is (5 times) greater than the impulse I_2 between the link and the chain above.

In a continuum limit, where the number of links becomes very large, using $N_2 = N_1/5$, and assuming no interference between the links, the equation of motion (5) becomes

$$dv/dt = \frac{v^2}{6(L-x)} + g, \tag{11}$$

with $\dot{x} = v$. We can contrast this solution with an energy conserving chain ($N_1 = N_2$), the fastest chain allowed by non-negative dissipation, where the governing equation is

$$dv/dt = \frac{v^2}{2(L-x)} + g. \tag{12}$$

In this energy conserving chain, as the chain falls down the initial mechanical energy gets concentrated into the chain section that is in the air, and ultimately to the last link. For $N_2 > 0$ the governing equation is generally singular as $x \to L$ because more and more energy is concentrated in a shorter and shorter length of chain, so the speed $v \to \infty$. In a discrete, energy-conserving chain, all of the initial energy is ultimately held as kinetic energy in the last link. A design which in principle conserves energy is shown in Fig. 3d. Making a bottom-pile design that approaches mechanical energy conservation is difficult in practice because of the dissipative nature of link collisions, link interference, slack in the strings, friction etc.

VI. EXPERIMENTS

A physical realization of the $N_2 = N_1/5$ chain of Figs. 3a and 4 is shown in Fig.5a. The experiments used two nominally identical chains as in Fig.5a for which the total lengths are 1.251m ± 2mm, and masses 218g ± 2g. The 25 links in each are cylindrical rods made of wood dowels and have average length of 10.5cm and diameter 1.25cm. They are inclined at an average angle of 13 deg with respect to horizontal and the mean centre to centre distance between consecutive links is 5.21cm. A thread made of unbraided Vectran (chosen for its tension stiffness) fibers holds the links together.

The top (horizontal) links of the two chains were dropped together from a height of 2.01m above the table, by a mechanical release (See Figs. 5b-5d). One chain falls onto the table while the other falls in the air, providing a clear picture for comparing the faster-than-gravity performance. This particular setup evolved to solve two problems.

1. **Air friction.** Our original idea was to drop, say, an apple and compare that to the falling chain.[29] But the air friction on an apple is different from that on the chain, so there would be a confounding effect.

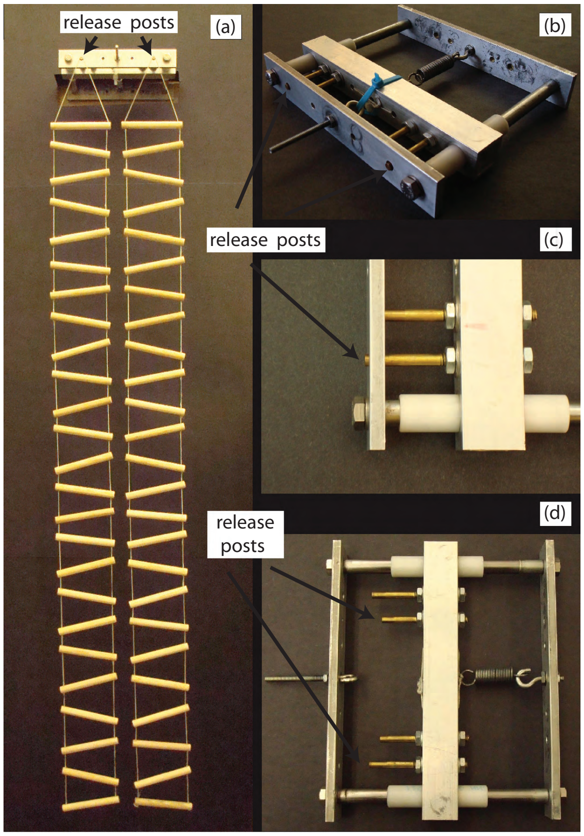

FIG. 5: Experimental apparatus (a) Two 'sucking' chains hanging from their simultaneous-release-mechanism. (b) Mechanism in zip-tied state with a loaded spring at the back. Chains are hung from two small (1.5 mm) posts protruding from the front. (c) A release post zoomed-in. (d) When the zip-tie is cut spring retracts the posts, releasing the chains 'simultaneously' (<3 ms).

2. **Elastic contraction.** When a chain is released the tension in the chain drops nominally to zero. Because the chain has some elasticity, this drop-to-zero starts an overall chain contraction that continues as the chain falls.[29] Use of Vectran reduced this contraction (the top of a nylon-string-based chain falls measurably faster than the Vectran chain). To eliminate this confounding effect, and the air-friction effect, we compared two simultaneously falling identical chains, one falling on a table (being sucked in to the table) and one falling freely next to the table.

Motion was filmed with Phantom V7.1 camera at 2000 fps. To make sure the chains were sufficiently similar we interchanged the chains and obtained the same results. Fig. 6, shows when the last link of the falling-on-the-table chain just hits its pile. It has won the race with the chain in air by almost 8 cm.

The experimental results are compared in Fig. 7 with previously derived theoretical models. Our central experimental result shows as the horizontal separation of the two points labeled as C. For our experimental chain, when each link collides with the table it pulls on the chain above it. Hence the chain is, in effect, pulled into the table it is falling on. The table which can only push up effectively sucks down. Towards the end, the chain acceleration is substantially more than g.

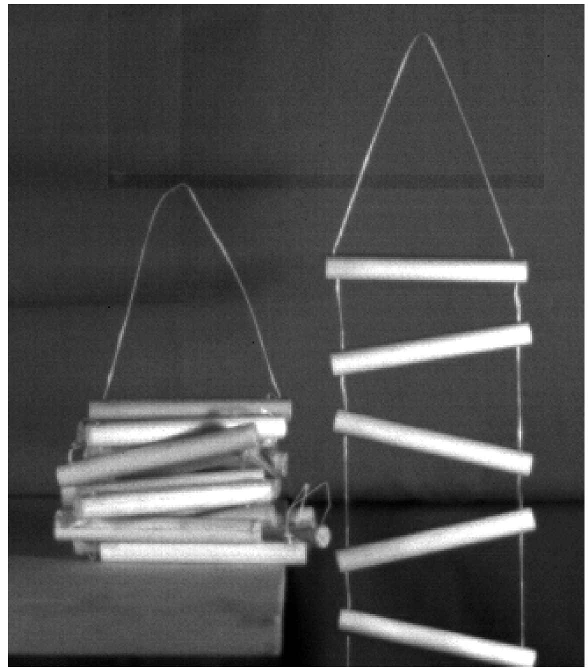

FIG. 6: The chain falling on the table counterintuitively wins the race when competing with the chain falling freely. At the instant shown (0.59s after release) the vertical distance between top links of chain on table and air is 7.6cm.

Videos of the experiments can be viewed at http://ruina.tam.cornell.edu/research/topics/fallingchains/.

VII. DISCUSSION

Consistent with all the experiments, the theory in the present paper shows that the simple chain problems are not well posed. Proper calculation depends on more information about the constitution of the chain. Different designs for links, their manner of falling, and the nature of the surface they fall on, generate different solutions.

At one theoretical extreme is the classic solution, where the chain falls with g and each link is slowed by the table ($N_2 = 0$). In the other theoretical extreme energy is conserved and the chain accelerates downwards much faster than g and each link is slowed equally by the table and the chain above ($N_2 = N_1$).

We have conclusively shown that the assumptions in the textbooks, regarding absence of interaction between the link hitting the ground and the chain above, are not universally valid.

To be fair, we used our set up to compare the falling of a pair of conventional metal chains with open oval links which easily disengage at collision. For such chains others have measured that the reaction force rises to $3mg$ as classically predicted[6–8]. Indeed, we could not detect any difference in the falling acceleration of the open-link chain falling freely and the open-link chain hitting the table. Thus classical open-link chains do seem to reasonably obey $N_2 = 0$.

Hamm and Geminard tested a ball chain, where links do not get disengaged at collision and also found an acceleration faster

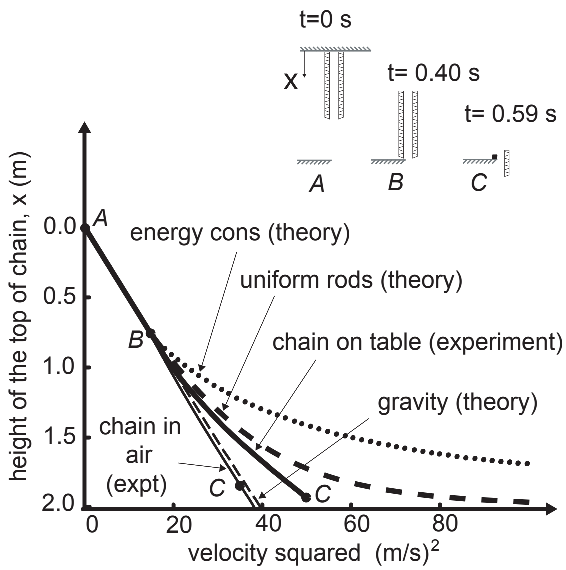

FIG. 7: Fall-distance x (from release to present instant) versus velocity squared for various types of chains. 'Gravity (theory)' is the falling inextensible chain with acceleration of g. 'Chain in air (expt)' is our free-falling chain, falling a bit slower than the gravity theory. 'Chain on table (experiment)' is our primary experimental result shown by the higher velocity at point C than for the point C with the chain in air. 'Uniform rods (theory)' is based on the $N_2 = N_1/5$ theory of our rod-based chain. 'Energy cons (theory)' is based on an $N_1 = N_2$ calculation and is theoretical upper limit for speed of a non-negative dissipation chain. All chains fall together until the bottom hits the table, or not.

than g.[29] They claim their contraction and air drag effects are negligible, so they did not need the side-by-side experiments. In their analysis the factor γ (what we would call $N_2/(N_1 + N_2)$) is estimated from experimental curves and they notice its dependence on the geometry at the fold.

The need for a constitutive law. For the U-chain Schagerl[18] implicitly points out that to calculate a motion, a constitutive relation for string material (continuum) or the chain links (discrete) is needed. Tomaszewski et. al,[23] also hint at this indirectly at the end of their paper "A falling rope exhibits even more interesting behavior because dissipation plays a more important role and elasticity becomes a crucial factor." McMillen[21] puts it directly: "without taking into account the constitutive relation, the rate of change of energy is not determined, which can lead to incorrect results". O'Reilly and Varadi[33] discuss the U-chain problem from a thermodynamic perspective; and with a model including a free parameter e (the constitutive parameter) they show the energy conservation and plastic impacts to be the two theoretical ends of the solution spectrum, as e varies from 0 ($N_2 = 0$) to 1 ($N_2 = N_1$). This paper extends these results to the bottom-pile chain.

Although these problems are generally considered theoretical exercises, the principles apply to systems of practical importance where a line, wire or chain is rolled in or unrolled, one example being a satellite antenna wire.[35]

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