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The buoyancy force on a particle in a rotating fluid is derived. We have not seen a derivation of this type of buoyancy force elsewhere although it is often presented *de facto*. The buoyancy force in hydrostatics is also rarely derived except for rectangular shapes. We derive it for a sphere.

The centrifuge is an instrument highly crucial to scientific research and industrial processes. All college and university science students have undoubtedly encountered the equations governing the motion of matter in a centrifugal field. In many textbooks and monographs covering the subject, the force on a particle in a centrifuge is blithely written without further comment as

$$F = V(\rho - \rho_f)\omega^2 R_0, \tag{1}$$

where V is the volume of the particle, ρ and ρ_f are the density of the particle and of the fluid medium, respectively, ω is the angular velocity of the centrifuge, and R_0 is the particle's distance from the central axis of the centrifuge. Most articles actually delete ρ_f from Eq. (1) and consequently ignore the "centrifugal buoyancy" force.[1]

The simplicity of this equation might lead one to believe that its derivation is simple and obvious. We found that this is not particularly so, when ρ_f is not negligible. Our derivation as presented here, could be used as an exercise for students or as a reminder that the obvious is not always so obvious.

The centrifugal force on a uniformly dense particle as experienced in the rotating frame of the fluid is

$$F_c = m\omega^2 R_0 = V\rho\omega^2 R_0 \tag{2}$$

as derived in numerous texts. This expression comprises part of the force in Eq. (1).

Due to the centrifugal force acting on the rotating fluid, there is a pressure gradient within the fluid that increases with radius. When this pressure acts on the particle, it experiences a force directed toward the central axis. In analogy with the hydrostatic pressure equation, the pressure difference dP across a small interval dr, at a distance r from the axis of rotation, is

$$dP = \rho_f \omega^2 r\, dr,$$

where the centrifugal acceleration $\omega^2 r$ replaces the acceleration of gravity. In order to determine the pressure within the rotating noncompressible fluid at a distance r from the center, it is necessary to integrate the pressure differences along a radius

$$P(r) = \int_0^r dP = \int_0^r \rho_f \omega^2 r\, dr = \tfrac{1}{2}\rho_f \omega^2 r^2. \tag{3}$$

Equation (3) describes pressure in the fluid as a function of the radius of the cylinder, the fluid density, and the angular velocity. Sommerfeld derived this same expression more formally.[2] However, he did not translate this pressure expression into a force on a particle. In order to do it, we need to integrate the pressure over the surface of the particle.

As in the calculations of Archimedean buoyancy, the derivation of the general solution is easier than is the surface integration for a specific shape (except perhaps for a cube). First we present the solution for a sphere and then do it for a generalized shape.

Figure 1 shows the coordinates for integration over a sphere, one of the simpler cases. Due to symmetry we need only consider the pressure in the x direction.

$$P_x = P\cos\theta = \tfrac{1}{2}\rho_f \omega^2 r^2 \cos\theta$$

and the force is

$$F = \int_{\text{surface}} P_x\, dA,$$

where

$$dA = a^2 \sin\theta\, d\theta\, d\phi$$

and

$$r^2 = R_0^2 + a^2 - 2aR_0\cos\theta - a^2\sin^2\theta\cos^2\phi,$$

giving

$$F = \int_0^{2\pi}\int_0^{\pi} \tfrac{1}{2}\rho_f\omega^2(R_0^2 + a^2 - 2aR_0\cos\theta - a^2\sin^2\theta\cos^2\phi)\cos\theta\, a^2\sin\theta\, d\theta\, d\phi.$$

Several terms in the integration can be immediately eliminated, as they are odd functions integrated over even intervals. The expression for the "buoyant" force reduces to

$$\begin{aligned} F &= -2\pi\rho_f\omega^2 a^3 R_0 \int_0^{\pi} \cos^2\theta\sin\theta\, d\theta \\ &= -2\pi\rho_f\omega^2 a^3 R_0(\tfrac{2}{3}) = -\tfrac{4}{3}\pi a^3\rho_f\omega^2 R_0 \\ &= -V\rho_f\omega^2 R_0. \end{aligned} \tag{4}$$

Combining this pseudobuoyant force with the centrifugal force, we get Eq. (1):

$$F = V(\rho - \rho_f)\omega^2 R_0.$$

We could have employed a hand-waving argument that is often used to justify Eq. (1). Namely, the buoyant force on the particle must equal the "weight" of the displaced water which is $\rho_f V\omega^2 R_0$. If this were not so, the fluid could never be in equilibrium.

To derive Eq. (1) in general, we take an arbitrary shape and put its center-of-mass on the x axis as shown in Fig. 2.

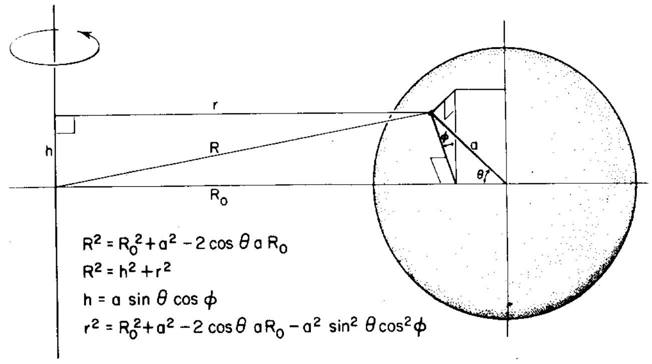

Fig. 1. Coordinates and relationships for integration over sphere in rotating fluid.

As the rotation is about the y axis, we need only consider a section of the particle in the x–z plane of some thickness dy. The other parallel sections of the particle will behave similarly. We then intersect the section by cuts parallel to the x–y plane dividing it into smaller sections.

First we will consider the section containing the center of mass. Due to symmetry all forces will cancel except those in the x direction. The pressure on each of the opposing faces, A_1 and A_2, is normal to the surface but, as we are only interested in the force in the x direction, we can consider the force on each face as its area projected onto the y–z plane times its respective pressure or

$$F_i = P_i A_i \cos\theta_i = P_i A,$$

where, by construction, the cross-sectional area is the same for both faces. Then the total force would be

$$F = \int (P_1 - P_2)dA = \int\int (P_1 - P_2)dy\, dz.$$

In this case

$$\begin{aligned} P_i &= \tfrac{1}{2}\rho_f\omega^2 x_i^2 \\ F &= \tfrac{1}{2}\rho_f\omega^2 \int\int (x_1^2 - x_2^2)dy\, dz \\ &= \rho_f\omega^2 \int\int\int x\, dx\, dy\, dz = \rho_f\omega^2 \int x\, dV. \end{aligned} \tag{5}$$

The radius to the center of mass of this section can be written

$$r_0 = \frac{\rho\int x\, dV}{\rho\int dV} = \frac{\int x\, dV}{V}.$$

Substituting into Eq. (5) we get, as expected,

$$F = \rho_f V\omega^2 r_0.$$

For a section above or below the x–y plane the calculation is slightly more complicated.

$$P_i = \tfrac{1}{2}\rho_f\omega^2 r_i^2 = \tfrac{1}{2}\rho_f\omega^2(x_i^2 + y_i^2).$$

Due to the particular construction of our sections, $y_1 = y_2$ for the two significant faces of all the sections. The rest proceeds as above.

$$\begin{aligned} F_i &= \tfrac{1}{2}\rho_f\omega^2 \int (r_1^2 - r_2^2)dA = \tfrac{1}{2}\rho_f\omega^2 \int (x_1^2 - x_2^2)dA \\ &= \rho_f\omega^2 \int x\, dV, \end{aligned}$$

and

$$F_i = \rho_f V_i\omega^2 r_i,$$

again in the x direction. Now r_i is the distance from the rotation axis to the center of mass of section i. The force in the z direction on a section above the center of mass is cancelled by an equal and opposite force on a section below the center of mass.

The total force on the particle is

$$F = \sum_i F_i = \rho_f\omega^2 \sum_i V_i r_i.$$

The radius to the center of mass of the entire particle, R_0, is

$$R_0 = \frac{\rho\Sigma V_i r_i}{\rho\Sigma V_i} = \frac{\Sigma V_i r_i}{V}$$

or

$$\Sigma V_i r_i = VR_0.$$

Substituting, we get Eq. (4):

$$F = V\rho_f\omega^2 R_0,$$

the centrifugal buoyancy force.

In addition to the forces in Eq. (1), the only other major force on the particle in a centrifuge would be the drag force as it moves through the fluid. It always opposes the motion, which according to Eq. (1) is radially out for particles denser than the fluid and radially in for particles less dense than the fluid. As is shown in a paper related to this one,[3] when the axis of rotation is held horizontal rather than vertical, at low Reynolds numbers the particles can go into quasistable orbits.

Another common result that is rarely presented to students is a derivation of Archimedes' principle. Most science students are aware of Archimedes' principle, but few of them have seen it demonstrated for an object other than a cube or a rectangular parallelepided or in some cases for a generalized object.[4] The computations for the buoyant force on a sphere immersed in a fluid are simpler than is the calculation already presented for centrifugal buoyant force, but the approach is similar. Everyone knows that the buoyant force equals the weight of the displaced water

$$F = V\rho_f g.$$

Actually calculating the force on a sphere or other shape would be a worthwhile exercise for the student. Figure 3 gives the coordinates and other symbols for the case of a sphere:

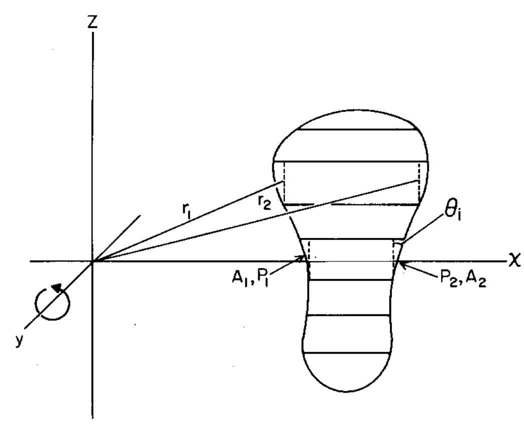

Fig. 2. Integration over particle of arbitrary shape in rotating fluid.

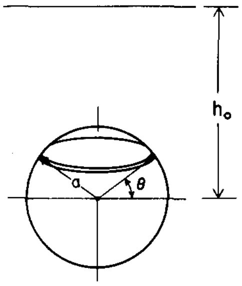

Fig. 3. Coordinates for integration over sphere in static fluid.

$$F = \int P\, dA,$$

where

$$dA = 2\pi a^2 \sin\theta\, d\theta,$$

$$P = \rho_f g h.$$

By symmetry we need only consider the z component

$$P_z = P \cos\theta, \qquad h = h_0 - a \sin\theta.$$

Thus,

$$F = 2\pi a^2 \int_{-\pi/2}^{\pi/2} \rho_f g (h_0 - a \sin\theta) \cos\theta \sin\theta\, d\theta.$$

Eliminating the odd term being integrated over this even interval,

$$F = 2\pi a^3 \rho_f g \int_{-\pi/2}^{\pi/2} \sin^2\theta (-\cos\theta)\, d\theta$$

$$= 2\pi a^3 \rho_f g (2/3) = 4/3 \pi a^3 \rho_f g,$$

$$F = V \rho_f g,$$

as anticipated.

ACKNOWLEDGMENT

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[1] T. Svedberg and K. O. Pederson, *The Ultracentrifuge* (Oxford, 1940), give the complete Eq. (1) and refer to a 1925 paper; T. Svedberg, Kolloid-Z., Zsigmondy-Festschrift, Erg. Bd. zu **36**, 53. This paper, however, did not give a derivation.

[2] A. Sommerfeld, *Mechanics of Deformable Bodies* (Academic, New York, 1950), p. 44.

[3] G. L. Wick, "Particles falling in a rotating fluid," Am. J. Phys. (to be published).

[4] See, for example, R. R. Long, *Mechanics of Solids and Fluids* (Prentice Hall, Englewood Cliffs, NJ, 1961), p. 124; I. Shames, *Mechanics of Fluids* (McGraw-Hill, New York, 1962), p. 48.