(Viscous Flow in Multiparticle Systems)

Motion of a Sphere and Fluid in a Cylindrical Tube

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Behavior of spherical particles suspended in fluids is of fundamental importance in problems involving settling, flow through packed beds, fluidization, and pneumatic conveying. In the investigations reported in these papers, uniform spherical particles are employed as an idealization of the systems found in practice. The following study is a mathematical treatment of the motion of a single sphere along the axis of a cylinder through which a viscous fluid is also moving. General expressions are derived for the force exerted on the sphere and the pressure drop experienced by the fluid. The application of these relationships to similar problems involved in assemblages of spherical particles is discussed. A simple relationship is derived for pressure drop through very dilute systems.

STUDIES described in these papers are part of a long range program that has as its ultimate objective a fundamental solution of the hydrodynamic relationships underlying low Reynolds number phenomena such as fluidization and sedimentation. These operations are often carried out in the Reynolds number range (based on particle diameter) of 5 or less and, as such, are amenable to mathematical treatment employing the hydrodynamic equations which describe the so-called creeping motion encountered at low Reynolds numbers.

In this case, as in most engineering applications, it is expedient to conceive simple models of the flow which lend themselves to analytical treatment but which, at the same time, furnish information of value concerning the more complex flow patterns encountered in practice. For these reasons, the sequence of theoretical investigations is supplemented by a corresponding sequence of experimental studies designed to justify the choice of idealized models and provide a source of data for verification of the results. In this manner, it is hoped to carry out a systematic study of the major variables encountered in the field of fluid-solids dynamics at low Reynolds numbers and thereby contribute to a greater understanding of such phenomena.

Mathematical analysis is difficult even in the case of a single particle, and further complications arise with assemblages of particles (*6*). As logical start toward solving some of the problems involved, a study of the influence of boundary proximity on the motion of a single sphere is described in this paper.

A theoretical interpretation of the phenomena of fluidization and hindered settling is usually complicated by factors such as particle agitation and rotation, mutual collision of the particles, interparticle bridging and, in general, a failure of the system to attain a steady state. The experimental investigation described in an accompanying paper (*7*) attempts to justify the choice of a fixed, cubical assemblage of uniform, spherical particles as an idealizing model of fluidized and sedimenting systems. This was accomplished by comparing the experimental data obtained on a series of fixed, cubic assemblages with the fluidization and sedimentation data of other investigators at equal values of the (particle) Reynolds number and fractional void volume.

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SINGLE SPHERE PROBLEM

In order to determine the magnitude of the effect of boundary proximity on the behavior of a spherical particle, a cylindrical boundary was selected. This choice was made because a cylindrical boundary can completely surround the fluid stream parallel to the direction of flow.

Relationships are developed for the general case of motion of a sphere along the longitudinal axis of a cylinder, through which fluid may also be moving. Expressions are given for both the resistance of the sphere and pressure drop caused by its presence. The equations are rigorous for the case of slow motion, in which the inertia terms of the equations of motion can be neglected. Hydrostatic and gravitational fields are not considered in the present solution. Approximations based on single sphere behavior, are derived for the case of dilute assemblages of particles.

To date, studies of the pattern of motion caused by a sphere moving through a fluid with a cylindrical boundary have been confined mostly to the case where the fluid itself is not initially in motion. McNown (*10*) recently published a survey of mathematical treatments in this field, together with some new experimental data at Reynolds numbers above the point where inertial effects are negligible. Ladenburg (*8*), in a theoretical study, derived an approximate expression for the effect of a cylindrical boundary on the behavior of a spherical particle falling in a quiescent fluid. Faxen (*4*), using somewhat the same method of derivation, extended the accuracy and corrected a numerical error in Ladenburg's treatment.

A very recent investigation by Wakiya (*14*) along similar lines treated the case of a stationary spherical obstacle in the flow of a viscous fluid through a tube. The work reported in this paper was completed before this study became available and furnishes independent confirmation of the results of Faxen and Wakiya by a different computation procedure as well as extension to the general case where both fluid and sphere move simultaneously, which is particularly relevant in the case of fluidization.

THEORETICAL DERIVATION

The basic equations which must be satisfied by the fluid are the continuity equation and the Stokes-Navier equations, from which the inertial terms have been discarded. For convenience, the origin of the coordinate system is taken at the center of the sphere. Cylindrical coordinates have been chosen to represent these equations because of the symmetry of the velocity field about the longitudinal axis of the cylinder. For the case of an incompressible fluid at steady state, when both the velocity of the fluid and sphere are low, the continuity and Stokes-Navier equations are

$$\frac{\partial u}{\partial X} + \frac{1}{R}\frac{\partial(R\, v_R)}{\partial R} = 0 \tag{1}$$

$$\frac{1}{\mu}\frac{\partial P}{\partial X} = \frac{\partial^2 u}{\partial X^2} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u}{\partial R}\right) \tag{2}$$

$$\frac{1}{\mu}\frac{\partial P}{\partial R} = \frac{\partial^2 v_R}{\partial X^2} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial v_R}{\partial R}\right) - \frac{v_R}{R^2} \tag{3}$$

The three boundary conditions which must be satisfied are:

At the surface of the sphere of radius a, where $r^2 = a^2 = X^2 + R^2$, there is no relative motion of fluid and sphere; hence,

$$u = 0 \tag{4}$$

$$v_R = 0 \tag{5}$$

At the cylinder wall where $R = R_0$, since there is no motion of the fluid with respect to the wall

$$u + U = 0 \tag{6}$$

$$v_R = 0 \tag{7}$$

At large distances from the sphere where $X = \pm\infty$, in the absence of the disturbing influence of the sphere, the field must be parabolic

$$u + U = U_0\left[1 - \left(\frac{R}{R_0}\right)^2\right] \tag{8a}$$

—i.e.,

$$u = u_0 = (U_0 - U) - BR^2 \tag{8}$$

and

$$v_R = (v_R)_0 = 0 \tag{9}$$

Thus, the simultaneous solution of equations 1, 2, and 3 for $u(X, R)$, $v(X,R)$, and $P(X,R)$ in the region $0 \leqslant X \leqslant \infty$ and $0 \leqslant R \leqslant R_0$, subject to the restrictions imposed by the boundary conditions of Equations 4 to 9, is required.

A technique for the approximate solution of a similar problem has been demonstrated by Ladenburg (*8*) and is termed the "method of reflections." Application of the technique consists of decomposing the velocity and pressure fields into a number of parts

$$u = \sum_{j=0}^{\infty} u_j = u_0 + u_1 + u_2 + u_3 + \ldots \tag{10}$$

$$v_R = \sum_{j=0}^{\infty} (v_R)_j = (v_R)_0 + (v_R)_1 + (v_R)_2 + (v_R)_3 + \ldots \tag{11}$$

$$P = \sum_{j=0}^{\infty} P_j = P_0 + P_1 + P_2 + P_3 + \ldots \tag{12}$$

in such a manner that each combination of terms u_j, $(v_R)_j$, P_j, is a particular solution of Equations 1, 2, and 3. Further, these particular solutions are so chosen that the following boundary conditions—applicable to the appropriate terms in Equations 10 and 11—are satisfied:

$$\text{At } r = a,\ u_1 = -u_0,\ (v_R)_1 = -(v_R)_0 \tag{13}$$

$$\text{At } R = R_0,\ u_2 = -u_1,\ (v_R)_2 = -(v_R)_1 \tag{14}$$

$$\text{At } r = a,\ u_3 = -u_2,\ (v_R)_3 = -(v_R)_2 \tag{15}$$

$$\text{At } R = R_0,\ u_4 = -u_3,\ (v_R)_4 = -(v_R)_3 \tag{16}$$

etc.

and at $X = \pm\infty$,

$$u_j = 0, \text{ for all } j \neq 0 \tag{17}$$

$$(v_R)_j = 0, \text{ for all } j \tag{18}$$

By utilizing these conditions, if the series represented by Equations 10 and 11 are cut off at any point, corresponding to an arbitrary degree of approximation, the original boundary conditions of Equations 5, 7, 8, and 9 will always be satisfied. In addition, Equation 4 will be satisfied if the series for u contains an even number of terms, while Equation 6 will be satisfied by an odd number of terms.

u_1, $(v_R)_1$ represents the "reflection" of the initial parabolic field, u_0, $(v_R)_0$, by the sphere. u_2, $(v_R)_2$ represents the reflection of u_1, $(v_R)_1$ at the cylinder wall. u_3, $(v_R)_3$ represents the reflection of u_2, $(v_R)_2$ by the sphere. This process can be repeated indefinitely until the contribution of succeeding terms to u is sufficiently small that the error introduced by the failure of a finite series exactly to satisfy either Equation 4 or 6 is vanishingly small.

Because of the symmetry of the velocity fields about the X- and R-axes, certain checks are available on the u_j and $(v_R)_j$ terms. Most notably, u_j must be an even function of both X and R, whereas $(v_R)_j$ must be an odd function of both X and R.

Simha (*11*), using the general solution of the continuity and Stokes-Navier equations available in terms of spherical harmonics (*9*), has determined the first reflection to be

$$u_1 = (U_0 - U)a\left[-\frac{3}{4r} - \frac{3X^2}{4r^3}\right] + (U_0 - U)a^3\left[-\frac{1}{4r^3} + \frac{3X^2}{4r^5}\right] + Ba^3\left[\frac{1}{2r} + \frac{X^2}{2r^3}\right] + Ba^5\left[\frac{1}{8r^3} + \frac{9X^2}{4r^5} - \frac{35X^4}{8r^7}\right] + Ba^7\left[\frac{3}{8r^5} - \frac{15X^2}{4r^7} + \frac{35X^4}{8r^9}\right] \tag{19}$$

$$(v_R)_1 = (U_0 - U)a\left[-\frac{3RX}{4r^3}\right] + U_0a^3\left[\frac{3RX}{4r^5}\right] + Ba^3\left[\frac{RX}{2r^3}\right] + Ba^5\left[\frac{11RX}{8r^5} - \frac{35RX^3}{8r^7}\right] + Ba^7\left[-\frac{15\,RX}{8r^7} + \frac{35\,RX^3}{8r^9}\right] \tag{20}$$

$$\frac{P_1}{\mu} = \frac{-3(U_0 - U)aX}{2r^3} + \frac{Ba^3X}{r^3} + \frac{21Ba^5X}{4r^5} - \frac{35Ba^5X^3}{4r^7} \tag{21}$$

Ladenburg has shown the next reflected field must satisfy the following relations:

$$u_2 = \frac{2}{\pi}\int_0^\infty\left[\frac{-i\lambda R}{2}(H + iG)J_1(i\lambda R) + H\,J_0(i\lambda R)\right]\cos\lambda X\,d\lambda \tag{22}$$

$$(v_R)_2 = \frac{2}{\pi}\int_0^\infty\left[\frac{\lambda R}{2}(H + iG)\,J_0(i\lambda R) - GJ_1(i\lambda R)\right]\sin\lambda X\,d\lambda \tag{23}$$

$$\frac{P_2}{\mu} = \frac{2}{\pi}\int_0^\infty \lambda(H + iG)\,J_0(i\lambda R)\sin\lambda X\,d\lambda \tag{24}$$

The functions, H and G, are independent of R and X and depend only on the parameter, λ. They must be evaluated from the boundary conditions of Equation 14. In order to bring the boundary conditions into the same coordinate system as Equations 22 and 23, Equations 19 and 20 must be expressed as integrals analogous to Equations 22 and 23. This is done by making use of the Bessel function integral of $1/r$

$$\frac{1}{r} = \frac{2}{\pi}\int_0^\infty K_0(i\lambda R)\cos\lambda X d\lambda \tag{25}$$

By repeated partial differentiation with respect to X and R, Equation 25 can be used to express all of the functions in Equations 19 and 20 as integrals. For example,

$$\frac{1}{2r} + \frac{X^2}{2r^3} = \frac{1}{r} + \frac{R}{2}\frac{\partial}{\partial R}\left(\frac{1}{r}\right) =$$

$$\frac{2}{\pi}\int_0^\infty \left[K_0(i\lambda R) - \frac{i\lambda R}{2}K_1(i\lambda R)\right]\cos\lambda X d\lambda \quad (26)$$

$$\frac{3}{8r^5} - \frac{15X^2}{4r^7} + \frac{35X^4}{8r^9} = \frac{\partial^4}{\partial X^4}\left(\frac{1}{r}\right) =$$

$$\frac{2}{\pi}\int_0^\infty \left[\frac{\lambda^4}{24}K_0(i\lambda R)\right]\cos\lambda X d\lambda \quad (27)$$

The Bessel functions, $K_0(i\lambda R)$ and $K_1(i\lambda R)$ are related by the equation

$$\frac{\partial}{\partial R}K_0(i\lambda R) = -i\lambda K_1(i\lambda R) \quad (28)$$

Equations 19 and 20 take the form

$$u_1 = \frac{2}{\pi}\int_0^\infty M\cos\lambda X d\lambda \quad (29)$$

$$(v_R)_1 = \frac{2}{\pi}\int_0^\infty N\sin\lambda X d\lambda \quad (30)$$

where $$M = \left[\frac{-3(U_0-U)a}{2} - \frac{(U_0-U)\lambda^2a^3}{4} + Ba^3 + Ba^5\lambda^2 + Ba^7\lambda^4\right]K_0(i\lambda R) - \left[\frac{-3(U_0-U)ai\lambda R}{4} + \frac{Ba^3i\lambda R}{2} + \frac{7Ba^5i\lambda^3R}{24}\right]K_1(i\lambda R) \quad (31)$$

and $$N = \left[\frac{-3(U_0-U)a\lambda R}{4} + \frac{Ba^3\lambda R}{2} + \frac{7Ba^5\lambda^3R}{24}\right]K_0(i\lambda R) - \left[\frac{-(U_0-U)a^3i\lambda^2}{4} + \frac{10Ba^5i\lambda^2}{24} + \frac{Ba^7i\lambda^4}{24}\right]K_1(i\lambda R) \quad (32)$$

By applying the boundary conditions at $R = R_0$ from Equations 22 and 23,

$$[M]_{R=R_0} = \frac{i\lambda R_0}{2}(H + iG)J_1(i\lambda R_0) - HJ_0(i\lambda R_0) \quad (33)$$

$$[N]_{R=R_0} = -\frac{\lambda R_0}{2}(H + iG)J_0(i\lambda R_0) + GJ_1(i\lambda R_0) \quad (34)$$

By substituting $R = R_0$ in Equations 31 and 32, equating Equation 31 to 33, and Equation 32 to 34, H and G are obtained by solving the two resulting equations simultaneously. For simplicity, the substitution, $\alpha = \lambda R_0$, has been made.

$$H = \left[\frac{K_0}{J_0}\left(3\{U_0-U\}a + \frac{\{U_0-U\}a^3\alpha^2}{4R_0^2} - Ba^3 - \frac{Ba^5\alpha^4}{R_0^2} - \frac{Ba^7\alpha^4}{24R_0^4}\right)\right] - \frac{1}{J_0^2 + J_1^2 - \frac{2J_0J_1}{i\alpha}}\left[\left(\frac{3\{U_0-U\}a}{4} - \frac{Ba^3}{2} - \frac{7Ba^5\alpha^2}{24R_0^2}\right) + \frac{J_1}{i\alpha J_0}\left(\frac{3\{U_0-U\}a}{2} + \frac{\{U_0-U\}a^3\alpha^2}{4R_0^2} - Ba^3 - \frac{Ba^5\alpha^2}{R_0^2} - \frac{Ba^7\alpha^4}{24R_0^4}\right)\right] \quad (35)$$

$$iG = \left[\frac{K_0}{J_0}\left(-\frac{\{U_0-U\}a^3\alpha^2}{4R_0^2} + \frac{5Ba^5\alpha^2}{12R_0^2} + \frac{Ba^7\alpha^4}{24}\right)\right] + \frac{1}{J_0^2 + J_1^2 - \frac{2J_0J_1}{i\alpha}}\left[\left(\frac{3\{U_0-U\}a}{4} - \frac{\{U_0-U\}a^3}{2R_0^2} - \frac{Ba^3}{2} + \frac{5Ba^5}{6R_0^2} - \frac{7Ba^5\alpha^2}{24R_0^2} + \frac{Ba^7\alpha^2}{12R_0^4}\right) + \frac{J_1}{i\alpha J_0}\left(-\frac{\{U_0-U\}a^3\alpha^2}{4R_0^2} + \frac{5Ba^5\alpha^2}{12R_0^2} + \frac{Ba^7\alpha^4}{24R_0^4}\right)\right] \quad (36)$$

For simplicity, the arguments of the Bessel functions have been dropped in the above equations.

$$J_0 = J_0(i\alpha) \quad (37)$$

$$J_1 = J_1(i\alpha) \quad (38)$$

$$K_0 = K_0(i\alpha) \quad (39)$$

$$K_1 = K_1(i\alpha) \quad (40)$$

Use has been made of the identity

$$J_0K_1 - J_1K_0 = \frac{1}{i\alpha} \quad (41)$$

Substitution of H and iG in Equations 22 and 23, followed by subsequent integration, yields the desired result for the velocity field "reflected" by the cylinder wall. The complex nature of certain of the functions to be integrated—e.g., $\frac{1}{J_0^2 + J_1^2 - \frac{2J_0J_1}{i\alpha}}$ — does not permit the integrals to be obtained in closed form. For this reason it is expedient to expand these functions by their infinite series representations. When the sphere diameter is small compared to that of the cylinder, in the neighborhood of the sphere—i.e., for $\frac{R}{R_0} \to 0$ and $\frac{X}{R_0} \to 0$

$$J_0(i\lambda R) = J_0\left(i\alpha\frac{R}{R_0}\right) = 1 + \frac{\alpha^2R^2}{4R_0^2} - \ldots. \quad (42)$$

$$J_1(i\lambda R) = J_1\left(i\alpha\frac{R}{R_0}\right) = \frac{i\alpha R}{2R_0} + \ldots. \quad (43)$$

$$\cos\lambda X = \cos\left(\frac{\alpha X}{R_0}\right) = 1 - \frac{\alpha^2X^2}{2R_0^2} + \ldots. \quad (44)$$

$$\sin\lambda X = \sin\left(\frac{\alpha X}{R_0}\right) = \frac{\alpha X}{R_0} + \ldots. \quad (45)$$

Substitution in Equations 22 and 23 gives

$$u_2 = \frac{2}{\pi R_0}\int_0^\infty [H]\,d\alpha + \frac{2R^2}{\pi R_0^3}\int_0^\infty\left[\frac{H\alpha^2}{2}\right]d\alpha + \frac{2R^2}{\pi R_0^3}\int_0^\infty\left[\frac{iG\alpha^2}{4}\right]d\alpha + \frac{2X^2}{\pi R_0^3}\int_0^\infty\left[\frac{-H\alpha^2}{2}\right]d\alpha + \ldots.. \quad (46)$$

$$(v_R)_2 = \frac{2RX}{\pi R_0^3}\int_0^\infty\left[\frac{H\alpha^2}{2}\right]d\alpha + \frac{2R^3X}{\pi R_0^5}\int_0^\infty\left[\frac{H\alpha^4}{8}\right]d\alpha + \ldots. \quad (47)$$

$(v_R)_2$ is an odd function with respect to X. Any integration over the sphere of the viscous stresses caused by this field will be equal to zero. Therefore, $(v_R)_2$ will not receive any further consideration. At the surface of the sphere, $a^2 = r^2 = R^2 + X^2$

$$[u_2]_{r=a} = \frac{2}{\pi R_0}\int_0^\infty H d\alpha - \frac{2a^2}{\pi R_0^3}\int_0^\infty H\alpha^2\,d\alpha - \frac{2X^2}{\pi R_0^3}\int_0^\infty\frac{H\alpha^2}{2}d\alpha + \frac{2R^2}{\pi R_0^3}\int_0^\infty\frac{iG\alpha^2}{4}d\alpha + \ldots. \quad (48)$$

The numerical integrals, $\int_0^\infty H d\alpha$; $\int_0^\infty H\alpha^2 d\alpha$; etc., can be broken down into integrals which have been evaluated by Faxen (*4*). These constants were also re-evaluated in the present investigation and result in the following equations in which numerical values are close to those of Faxen.

$$[u_2]_{r=a} = 2.105\frac{a}{R_0}[(U_0 - U) - Ba^3] - 2.179\frac{a^3}{R_0^3}[U_0 - U] - 0.138\frac{aR^2}{R_0^3}[U_0 - U] + \dots \quad (49)$$

In order to obtain the next reflection Equation 15, at $r = a$, $u_3 = -u_2$; $(v_R)_3 = -(v_R)_2$, would be used.

However, $[u_2]_{r=a}$ is a parabolic field analogous to $[u_0]_{r=a}$. Therefore, u_3, $(v_R)_3$ bear the same relation to $[u_2]_{r=a}$ as u_1, $(v_R)_1$ bear to $[u_0]_{r=a}$. It is, therefore, easily possible to repeat the reflection technique indefinitely until the magnitude of the reflected fields becomes insignificant.

Simha (*11*) has determined the frictional resistance, W, of the sphere due to the initial parabolic field and the first reflection.

$$W_0 + W_1 = 6\pi\mu a(U_0 - U) - 4\pi\mu Ba^3 \quad (50)$$

From Equation 49

$$W_2 + W_3 = 6\pi\mu a(U_0 - U)\left[2.105\frac{a}{R_0} - 2.087\frac{a^3}{R_0^3}\right] - 4\pi\mu Ba^3\left[2.105\frac{a}{R_0}\right] \quad (51)$$

If the procedure is repeated for higher order reflections, the sum of the geometric series obtained is

$$W = \frac{6\pi\mu a(U_0 - U) - 4\pi\mu Ba^3}{\left[1 - 2.105\left(\frac{a}{R_0}\right) + 2.087\left(\frac{a}{R_0}\right)^3\right]} \quad (52)$$

If there is no motion of the sphere with respect to the wall, $U = 0$ and

$$W = \frac{6\pi\mu\, aU_0\left[1 - \frac{2}{3}\left(\frac{a}{R_0}\right)^2\right]}{\left[1 - 2.105\left(\frac{a}{R_0}\right) + 2.087\left(\frac{a}{R_0}\right)^3\right]} \quad (53)$$

The reflected pressure, P_2, can be obtained by substitution in Equation 24. At the wall, $R = R_0$ and

$$\frac{P_2}{\mu} = \frac{2}{\pi}\int_0^\infty\left[\frac{3(U_0 - U)a}{2} - Ba^3 - \frac{7Ba^5\lambda^2}{12}\right]\lambda K_0(i\lambda R_0)\sin\lambda X\, d\lambda$$
$$+\frac{2}{\pi}\int_0^\infty\left[\frac{3(U_0 - U)a}{2} - Ba^3 - \frac{7Ba^5\lambda^2}{12}\right]\frac{J_1(i\lambda R_0)\sin\lambda X d\lambda}{iR_0\left[J_0^2(i\lambda R_0) + J_1^2(i\lambda R_0) - \frac{2J_0J_1}{i\lambda R_0}\right]}$$
$$+\frac{2}{\pi}\int_0^\infty\left[\frac{-(U_0 - U)a^3}{2R_0^2} + \frac{5Ba^5}{6R_0^2} + \frac{Ba^7\lambda^2}{12R_0^2}\right]\frac{\lambda J_0(i\lambda R_0)\sin\lambda X d\lambda}{\left[J_0^2(i\lambda R_0) + J_1^2(i\lambda R_0) - \frac{2J_0J_1}{i\lambda R_0}\right]} \quad (54)$$

The first term of Equation 54 is equal to $-P_1/\mu$. To find the effect of the sphere on the pressure drop in the tube, P_2 must be evaluated at a great distance from the sphere.

Evaluation of Equation 54 at $X = \pm\infty$ is accomplished by expanding the Bessel functions in series. It is found that, with the limits taken, all the terms in these series vanish except those involving the integral of $\frac{\sin\lambda X\, d\lambda}{\lambda}$, and the integrations can be performed exactly to give the following result. Evaluation of Equation 54 at $X = \pm\infty$ yields

$$\left[\frac{P_1 + P_2}{\mu}\right]_{X=\pm\infty} = \mp\frac{6a(U_0 - U)\left(1 - \frac{2a^2}{3R_0^2}\right) - 4Ba^3\left(1 - \frac{5a^2}{3R_0^2}\right)}{R_0^2} \quad (55)$$

If the higher order reflections are taken into account

$$\left[\frac{R_0^2 P_2'}{\mu}\right]_{X=\pm\infty} = \mp\frac{6a(U_0 - U)\left(1 - \frac{2a^2}{3R_0^2}\right) - 4Ba^3\left(1 - \frac{5a^2}{3R_0^2}\right)}{1 - 2.105\frac{a}{R_0} + 2.087\frac{a^3}{R_0^3}} \quad (56)$$

Now, $\Delta P' = [P']_{X=-\infty} - [P']_{X=+\infty}$

Therefore,

$$\Delta P' = \mu\frac{\left[12a(U_0 - U)\left(1 - \frac{2a^2}{3R_0^2}\right) - 8Ba^3\left(1 - \frac{5a^2}{3R_0^2}\right)\right]}{R_0^2\left[1 - 2.105\frac{a}{R_0} + 2.087\frac{a^3}{R_0^3}\right]} \quad (57)$$

If F is designated as the total force required to maintain flow, $F = (\Delta P')(\pi R_0^2)$, in addition to that required to maintain normal Poiseuille flow, and powers of a/R_0 above the third are neglected, the following expression relates F and W:

$$F = 2W\left[1 - \frac{2}{3}\left(\frac{a}{R_0}\right)^2\right] \quad (58)$$

When there is no motion of the sphere with respect to the wall, $U = 0$

$$\Delta P' = \frac{12a\mu U_0\left[1 - \frac{4a^2}{3R_0^2} + \frac{10a^4}{9R_0^4}\right]}{R_0^2\left[1 - 2.105\frac{a}{R_0} + 2.087\frac{a^3}{R_0^3}\right]} \quad (59)$$

FORCE ACTING ON SPHERE

Equation 52 can be applied to any situation where it is desired to predict the force exerted on the sphere alone. If, as is the case in sedimentation problems, the force acting on the sphere is balanced by the action of gravity, the sphere will attain a constant terminal velocity corresponding to the equalization of the drag force exerted by fluid motion and the gravitational force corrected for buoyancy. For cases of viscous flow, the correction applicable to terminal settling velocity is thus the same magnitude as that for drag force as evaluated above.

In the ideal case, with a sphere falling at a uniform speed through a stationary fluid extending to infinity or with the same fluid moving at uniform velocity past a stationary sphere, the familiar Stokes' law applies. Table I gives the effect of a cylindrical boundary on the force exerted on a sphere for the limiting cases—stationary fluid, $U_0 = 0$; stationary sphere, $U = 0$; no slip velocity, $U = U_0$ (the slip velocity is defined as $U_0 - U$).

For values of $a/R_0 > 0.3$ it would be necessary to employ terms with higher powers than those in $(a/R_0)^3$ appearing in Equation 52. For the case of a sphere falling in a stationary fluid, available experimental data (*1, 5, 10*) indicate good agreement with Table I.

Faxen (*4*) and Wakiya (*14*) carry the approximation for a stationary fluid to $(a/R_0)^5$. If their formula is applied to the case for $a/R_0 = 0.5$, a value for the correction factor roughly 10% higher than experimental values of McNown (*10*) and Francis (*5*) is obtained. It seems likely that convergence is not so satisfactory as higher ratios of a/R_0 are involved.

The negative signs in the second and fourth columns of Table I indicate a force on the sphere opposite to the direction taken as positive for either fluid or sphere. Thus, if a sphere moves upward in a

Table I. Comparison of Force Exerted on a Sphere with Stokes' Law Resistance

a/R_0	Stationary Fluid $\frac{W}{-6\pi\mu a U}$	Stationary Sphere $\frac{W}{6\pi\mu a U_0}$	Zero Slip $\frac{W}{-6\pi\mu a U_0}$
0.005	1.01	1.01	...
0.01	1.02	1.02	...
0.05	1.12	1.12	...
0.10	1.26	1.25	0.008
0.20	1.68	1.63	0.045
0.30	2.36	2.22	0.142

stationary fluid the force exerted on it will be downward. At values of a/R_0 less than 0.1, the magnitude of the correction to Stokes' law is the same whether the sphere or fluid moves. Above $a/R_0 = 0.1$, terms in a/R_0 of higher power become significant, and there is an appreciable difference in resistance whether the fluid or the sphere remains stationary.

The case for zero slip indicates no significant force on the sphere until a/R_0 exceeds 0.1. At higher values of a/R_0, zero slip is possible only if an additional force is applied to the sphere in the same direction as the motion, $U_0 = U$. Thus, if flow is upward and gravity supplies the external force, the particle must have a specific gravity lower than that of the surrounding fluid.

For the case where $W = 0$—i.e., fluid and sphere of the same specific gravity

$$\frac{U_0 - U}{U} = \frac{2}{3}\left(\frac{a}{R_0}\right)^2$$

and for $a/R_0 = 0.3$, $\frac{U_0 - U}{U} \times 100 = 6\%$ slip, referred to the axial velocity of sphere and undisturbed fluid. This slip velocity is due to the fact that the sphere cannot accommodate itself perfectly to a Poiseuille field as it could if the flow were uniform or purely rotational, in which case no such effect would occur.

TOTAL RESISTANCE TO FLOW

In engineering applications the pressure necessary to maintain flow through a system is important. Equation 58 enables the prediction of the incremental pressure above that required to maintain the original undisturbed flow. Stokes' law, since it applies to a fluid which extends to infinity, is not useful in obtaining pressure drop. It might be guessed that, in a system where the flow is bounded parallel to the direction of flow, a total additional force equal to that indicated by Stokes' law could be applied. Surprisingly, this is not the case and, in fact, the total force required is exactly twice that predicted from Stokes' law, if the walls of the containing cylinder are at an infinite distance from the sphere. Thus, from Equation 58, letting a/R_0 approach zero

$$F = 12\pi\, a\mu(U_0 - U) \tag{60}$$

Table II gives numerical values for the effect of a cylindrical boundary on total resistance due to the presence of a sphere at the axis of a tube for the same limiting cases as Table I—namely, stationary fluid, $U_0 = 0$; stationary sphere, $U = 0$; zero slip, $U = U_0$.

Table II. Comparison of Total Resistance Due to Sphere with Stokes' Analog

$\frac{a}{R_0}$	Stationary Fluid $\frac{F}{-12\pi\mu a U}$	Stationary Sphere $\frac{F}{12\pi\mu\, a U_0}$	Zero Slip $\frac{F}{-12\pi\, a\mu U_0}$
0.005	1.01	1.00	...
0.01	1.02	1.02	...
0.05	1.12	1.12	...
0.10	1.25	1.24	0.008
0.20	1.63	1.59	0.043
0.30	2.22	2.08	0.133

For values of $a/R_0 > 0.3$ it would be necessary to employ terms with higher powers in a/R_0 in Equation 58, and a more complicated over-all expression would result (*14*). For values of a/R_0 less than 0.1 it is possible to employ the same correction factor and, furthermore, the correction factor is the same one that applies to the force exerted on the sphere alone. The second column of Table II has the same numerical values as the third column of Table I.

The condition for no pressure drop corresponds to the case where $W = 0$—i.e., with fluid and sphere of the same specific gravity. Since there is no pressure drop, no additional energy will be consumed by friction due to this slip at steady-state conditions, above that required if the space were occupied by the fluid instead of the sphere.

ASSEMBLAGES OF SPHERES

For the case of dilute assemblages, where interaction and wall effect terms involving a/R_0 and its powers can be neglected, the force exerted on each of the spheres will be given by Stokes' law. The rate of sedimentation, or velocity required for fluidization, can be computed on this basis. For the calculation of pressure drop a corresponding simple derivation is possible, if it is assumed that the same force will be exerted on spheres not at the axis for a given approach velocity as the force indicated by Equation 59 for a concentrically placed sphere. If it is assumed for simplicity that the assemblage is not moving, $U = 0$, the pressure drop due to the presence of a single sphere is

$$\Delta P' = \frac{F}{\pi R_0^2} = \frac{12\pi\mu a U_0}{\pi R_0^2} = \frac{12\mu a U_0}{R_0^2} \tag{61}$$

If there are n spheres involved, the total pressure drop is

$$\Delta P_S = \frac{12 n\mu a V}{R_0^2} \tag{62}$$

where V is the average superficial velocity of the fluid with respect to the pipe.

If ϵ = void volume fraction and L = the length of the assemblage along the direction of flow,

$$n = \frac{3\, R_0^2 L\,(1 - \epsilon)}{4a^3} \tag{63}$$

Therefore,

$$\Delta P_S = 9(1 - \epsilon)\,\frac{V\mu L}{a^2} \tag{64}$$

This equation is only an approximation, in view of the assumptions made in its derivation. However, it does indicate that the pressure drop through very dilute fluidized assemblages should approach a value double that required to simply balance the buoyant force exerted on the spheres. At present there are not sufficient accurate data on dilute systems to test this relationship.

No rigorous treatment exists for estimation of the effect of intereaction between spheres and a containing wall that will occur as assemblages become more concentrated. The classical study by Smoluchowski (*12*) considered the case of a number of spheres suspended in an infinite medium. A method was presented for calculation of the force exerted on each sphere, and the case for two spheres was worked out in detail to a first approximation. It appears that, as the number of spheres increases, the force exerted on each sphere decreases progressively. In cases involving a boundary this is not usually so, and in situations where pressure drop occurs a boundary, enclosing the flowing stream, is always present.

Several interesting theoretical studies have been made for the case of viscous flow around uniform spheres (*2, 3, 13*). These studies, for simplification, assumed that the effects due to boundary walls are small compared with interaction among spheres. They allowed for interaction by approximate solutions of the equations of motion which should best apply in dilute assemblages. Thus, the authors are confronted by the paradox that, in the range of concentration where the interaction effects can best be computed, it is also necessary to consider the effect of bounding walls—i.e., in very dilute systems. It should be possible to extend the present treatment to assemblages of several spheres to indicate the relative importance of these interaction effects.

SUMMARY

Expressions are derived, based on the Stokes-Navier equations of motion and neglecting inertia terms, for the velocity field produced by the presence of a sphere at the axis of a cylindrical tube through which fluid is passing. The results are given in general equations expressing the force exerted on the sphere and on the cylindrical boundary in terms of appropriate independent variables.

Numerical applications for the case of a single sphere are given that make possible the prediction of rate of movement of the sphere and pressure drop, as well as the extent to which correction factors to the simple Stokes' law type of equations are necessary. It is demonstrated that a sphere falling in a fluid at a given velocity does not experience the same resistance as that caused by fluid moving with the same axial velocity past a stationary sphere. In addition, the pressure drop caused by the presence of a sphere is not simply equivalent to the buoyant force exerted on it. The application of the relationships to the similar problems involved in a spherical assemblage is also discussed. A simple relationship is derived for pressure drop through very dilute systems.

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NOMENCLATURE

(Consistent absolute units)

a = radius of sphere

B = U_0/R_0^2

F = total force necessary to maintain flow due to presence of sphere
$= (\Delta P')(\pi R_0^2)$

G, H = functions of parameter λ

$J_0(iy)$ = Bessel function of the first kind, zero order, with imaginary argument

$$= \sum_{m=0}^{\infty} \frac{y^{2m}}{2^{2m}(m!)^2}$$

$J_1(iy)$ = Bessel function of the first kind, first order, with imaginary argument

$$= \frac{iy}{2}\sum_{m=0}^{\infty} \frac{y^{2m}}{2^{2m}m!(m+1)!}$$

$K_0(iy)$ = Bessel function of the second kind, zero order, with imaginary argument

$$= -J_0(iy)\left[\ln\frac{y}{2} + \gamma\right] + 2\sum_{m=1}^{\infty}\left(\frac{y^{2m}}{2^m}\right)\sum_{k=0}^{\infty}\left(\frac{y^{2k}}{2^{2k}k!(2m+k)!}\right)$$

Note. These Bessel functions of the second kind, K_n, are related to Neumann's Bessel functions of the second kind, Y_n, by the relation

$$K_n = -Y_n + J_n\left(\ln 2 - \gamma + \frac{i\pi}{2}\right)$$

for both real and imaginary arguments, and for all orders $n \geqslant 0$.

γ = Gauss constant = 0.5772157

$K_1(iy)$ = Bessel function of the second kind, first order, with imaginary argument

L = length of assemblage measured in direction of flow

M, N = functions defined by Equations 31 and 32

n = number of spheres in an assemblage

P = pressure

P' = increase in pressure due to presence of sphere or spheres

$\Delta P'$ = pressure drop due to presence of a single sphere

ΔP_S = total pressure drop due to presence of an assemblage of spheres

P_1, P_2, P_3, etc. = pressure due to velocity fields u_1, u_2, u_3, etc., respectively

r = radial distance from center of sphere to element of fluid

R = perpendicular distance from longitudinal axis of cylinder to element of fluid

U = velocity of sphere in direction of X-positive, with respect to a transverse plane passing through arbitrary point on pipe wall

U_0 = velocity of element of fluid at longitudinal axis of cylinder, in $+X$-direction, at large distance from sphere

u = relative velocity of element of fluid in $+X$-direction, with respect to center of sphere

v_R = velocity of element of fluid in direction perpendicular to longitudinal axis of cylinder (radial direction)

V = average fluid velocity in empty tube approaching an assemblage

W = frictional resistance of sphere

X = distance from center of sphere along longitudinal axis of cylinder to element of fluid

α = λR_0 = parameter

ϵ = fractional void volume

λ = parameter

μ = viscosity

LITERATURE CITED

(1) Bacon, L. R., *J. Franklin Inst.*, **221**, 251 (1936).
(2) Brinkman, H. C., *Appl. Sci. Research*, **A1**, 27 (1947); *Proc. Koninkl. Ned. Akad. Wetenschap*, **50**, 618, 860 (1947).
(3) Burgers, J. M., *Proc. Koninkl. Akad. Wetenschap. Amsterdam*, **43**, 315, 425, 645 (1940); **44**, 1177 1945, (1941); **45**, 9, 126 (1942).
(4) Faxen, H., *Arkiv. Mat., Astron. Fysik*, **17** (1923); dissertation, Uppsala University, 1921.
(5) Francis, A. W., *Physics*, **4**, 403 (1953).
(6) Happel, J., Ind. Eng. Chem., **41**, 1161 (1949).
(7) Happel, J., and Epstein, N., *Ibid.*, **46**, 1187 (1954).
(8) Ladenburg, R., *Ann. phys.*, **23**, 447 (1907); dissertation, University of Munich, Barth, 1907.
(9) Lamb, H., "Hydrodynamics," 6th ed., p. 594, Cambridge, England, University Press, 1932.
(10) McNown, J. S., Lee, H. M., McPherson, M. B., and Engez, S. M., *Proc. VII Intern. Congr. Appl. Mech., London* (1948); State Univ. Iowa, Reprints in Eng., *Reprint* **81**.
(11) Simha, R., *Kolloid-Z.*, **76**, 16 (1936).
(12) Smoluchowski, M. M., *Bull. Acad. Sci. Cracovie*, **1A** (1911).
(13) Uchida, S., *Rept. Inst. Sci. and Technol., Univ. Tokyo*, **3**, 97 (1949); abstract, Ind. Eng. Chem., **46**, 1194 (1954).
(14) Wakiya, S., *J. Phys. Soc. Japan*, **8**, 254 (1953).

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(*Continued*)