The Motion of a Sphere in a Rotating Liquid.

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In some recent papers* the author has drawn attention to certain general properties of rotating fluids, especially to the differences which may be expected between two- and three-dimensional motion. Unfortunately, mathematical difficulties have so far prevented the solution of any three-dimensional problem in a rotationally moving fluid from being obtained, except in one case, when the motion is very slow. In this case, Prof. Proudman has shown how it is possible to approximate to the solution of the problem of the slow motion of a sphere in a rotating fluid.† Even in this case the analysis is very complicated.

There seems little prospect of obtaining a more general solution of the problem when the inertia terms which Proudman neglected are taken into account. On the other hand, it is shown in the following pages that a solution can be obtained in the case when the sphere moves *steadily* along the axis of rotation of the fluid. The limitation imposed by considering only a steady motion necessarily excludes the case considered by Proudman, for all slow steady motions of a rotating fluid are two-dimensional.

In the case when the sphere moves steadily with velocity U along the axis of a fluid rotating with angular velocity Ω, it is possible to reduce the flow to a steady motion by superposing a velocity $-\mathrm{U}$ on the whole system. Since the motion is symmetrical about the axis, only two independent co-ordinates specifying position are necessary, namely, r, the distance of any point from the centre of the sphere, and θ, the angle between the radius from the centre and the axis of symmetry.

Let u be the component of velocity of the fluid along a radius from the centre of the sphere, v the component in an axial plane and perpendicular to the radius, w the component perpendicular to the axial plane. The scheme is shown in fig. 1. Since the motion is symmetrical, the equation of continuity is satisfied if

$$u = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad v = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, \tag{1}$$

where ψ is Stokes' stream function.

* 'Roy. Soc. Proc.,' A, vol. 100, p. 114 (1921); 'Proc. Camb. Phil. Soc.,' vol. 20, p. 326 (1921); 'Roy. Soc. Proc.,' A, vol. 93, p. 99 (1917).

† 'Roy. Soc. Proc.,' A, vol. 92, p. 408 (1916). Proudman's work has recently been extended by Mr. S. F. Grace (these 'Proceedings,' p. 89, *supra*).

Since the motion is symmetrical, the equations of motion are

$$\frac{\mathrm{D}u}{\mathrm{D}t}-\frac{v^2}{r}-\frac{u^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t}-\frac{w^2\cot\theta}{r}+\frac{uv}{r}=-\frac{1}{\rho r}\frac{\partial p}{\partial\theta}, \tag{3}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t}+\frac{uw}{r}+\frac{vw\cot\theta}{r}=0, \tag{4}$$

where p is the pressure and ρ the density of the fluid.

Since the motion is steady $\frac{\mathrm{D}}{\mathrm{D}t}$ is $u\frac{\partial}{\partial r}+\frac{v}{r}\frac{\partial}{\partial\theta}$.

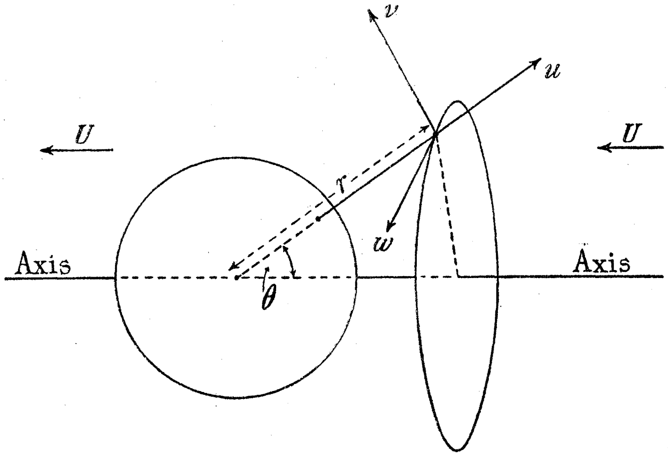

Fig. 1.—Scheme of co-ordinates.

It is easy to verify that $w = \mathrm{A}\psi/(r\sin\theta)$ satisfies the equation (4), and if the motion at infinity consists of a flow with uniform velocity $-\mathrm{U}$ parallel to axis and a rotation about this axis with angular velocity Ω, the constant A is evidently equal to $2\Omega/\mathrm{U}$, so that

$$w = 2\Omega\psi/(\mathrm{U}r\sin\theta). \tag{5}$$

This equation evidently expresses the fact that the circulation in a ring of fluid, which is symmetrical with respect to the axis, remains constant during the motion.

Let us now search for possible solutions of the form

$$\psi = f\sin^2\theta, \tag{6}$$

where f is a function of r only.

The components of velocity are then

$$u = -\frac{2f\cos\theta}{r}, \qquad v = \frac{f'\sin\theta}{r}, \qquad w = \frac{2\Omega}{\mathrm{U}}\frac{f\sin\theta}{r}, \tag{7}$$

where f' is df/dr.

Eliminating p between (2) and (3), it will be found that the form assumed for ψ is legitimate, provided that f satisfies the equation

$$r^3f''' - 2r^2f'' - 2rf' + 8f + (4\Omega^2/\mathrm{U}^2)(rf' - 2f) = 0. \tag{8}$$

The complete solution of this equation is

$$f = \mathrm{C}z^2 + \mathrm{A}\left(\cos z - \frac{\sin z}{z}\right) + \mathrm{B}\left(\sin z + \frac{\cos z}{z}\right),$$

or alternatively

$$f = \mathrm{C}z^2 + \mathrm{D}\left\{\cos(z+\epsilon) - \frac{\sin(z+\epsilon)}{z}\right\}, \tag{9}$$

where

$$k = 2\Omega/\mathrm{U}, \qquad z = kr,$$

and A, B, C, D, ϵ are arbitrary constants. When z becomes infinite, the first term in (9) becomes large compared with the others. The motion at a great distance from the sphere is therefore represented by $f = \mathrm{C}z^2$, and, on comparing this with (7), it will be seen that this represents a uniformly rotating fluid moving with velocity $-2\mathrm{C}k^2$ along the axis. Hence

$$\mathrm{C} = \mathrm{U}/2k^2. \tag{10}$$

The condition at the surface of the sphere, $r = a$, is $u = 0$. Writing $\mu = ka$, this condition gives

$$\tfrac{1}{2}\mathrm{U}\frac{\mu^2}{k^2} + \mathrm{D}\left\{\cos(\mu+\epsilon) - \frac{\sin(\mu+\epsilon)}{\mu}\right\} = 0. \tag{11}$$

Any values of D and ϵ which satisfy (11) lead to a possible solution of the problem.

It appears, therefore, that there are an infinite number of possible motions round a sphere moving steadily along the axis of a rotating fluid, and that all of them vanish at infinity.

I have not been able to discover how the motion could be set up. Perhaps the different solutions represent the stream-lines due to different ways of starting the motion. It is clear, however, that some of the possible motions represented by (9) could not be set up by starting the sphere from rest; it would be impossible for instance to set up a motion in this way for which f vanished or changed sign anywhere except at the surface of the sphere, because a negative value of f corresponds with a reversed rotation of the fluid about the axis. Such a motion is dynamically possible however, and it corresponds with a case in which all the liquid inside a certain sphere, concentric with the solid sphere, moves with it, forming a kind of sheath of liquid which possesses a rotation about the axis opposite to the rotation at infinity.

It is possible that these considerations may be extended so as to differentiate between the various possible motions corresponding to the various possible pairs of values of D and ϵ which are consistent with (11). The circulation round any symmetrical ring of fluid remains constant during any motion of the fluid. It is equal to $I = 2\pi\Omega y^2{}_0$, where y_0 is the initial distance of a particle of the ring considered from the axis. If the sphere starts from rest, the total deficiency of fluid in the field which possesses circulation lying between I and $I+(dI/dy_0)\delta y_0$ below that which the fluid would possess in the absence of the sphere, is the volume of fluid displaced by the part of the sphere which is contained between cylinders concentric with the axis, whose radii are y_0 and $y_0+\delta y_0$. This volume is

$$2\pi y_0 (a^2-y_0{}^2)^{\frac{1}{2}}\,\delta y_0 \text{ if } y_0 < a, \quad \text{or} \quad 0 \text{ if } y_0 > a.$$

When the sphere moves in steady motion with velocity U, ψ is connected with y_0 by the equation

$$y_0{}^2 = 2\psi/U.$$

The total deficiency of fluid possessing circulation lying between I and $I+(dI/dy_0)\delta y_0$ is, therefore, found by calculating the deficiency of fluid possessing circulation between I and $I+(dI/d\psi)\delta\psi$ (where $\delta\psi = Uy_0\delta y_0$), by integrating the total volume of fluid between the stream-lines ψ and $\psi+\delta\psi$ through the whole field. If this deficiency is not equal to $2\pi(a^2-y_0{}^2)\,\delta y$, when $y_0<a$ and 0 when $y_0>a$, then the motion could not be produced by starting a sphere from rest in a rotating fluid, unless there is a finite motion of the fluid at infinity along the stream-lines close to the axis during the time the motion is being established. This reasoning makes it appear that there is very little chance that any of the motions represented by (9) would be started by moving a sphere from rest in a rotating fluid.

It is interesting to notice that it is possible to find solutions in which $u = v = w = 0$ at the surface of the sphere, so that there is no slipping between the fluid and the surface of the sphere. The condition $U = 0$ at $r = a$ leads to the equation $0 = [f']_{z=\mu}$,

or
$$\frac{\mu U}{k^2}+D\left\{\left(\frac{1}{\mu^2}-1\right)\sin(\mu+\epsilon)-\frac{1}{\mu}\cos(\mu+\epsilon)\right\} = 0. \tag{12}$$

This together with (11) yields the following values for D and ϵ.

$$D = \tfrac{1}{2}U(\mu^4+3\mu^2+9)/k^2, \tag{13}$$

$$\tan(\mu+\epsilon) = 3\mu(3-\mu^2)^{-1}, \tag{14}$$

so that the motion represented by

$$\frac{2k^2}{U}f = z^2+(\mu^4+3\mu^2+9)^{\frac{1}{2}}\left\{\cos(z+\epsilon)-\frac{\sin(z+\epsilon)}{z}\right\}, \tag{15}$$

is one for which the disturbance due to the sphere vanishes at infinity, and it is characterised by the fact that there is no slipping at the surface of the sphere.

This may be a point of some importance because it is the assumption that there is slipping at the surface of a solid body moving in a liquid which vitiates all the ordinary hydrodynamical theories of the motion of solids in fluids. It is possible, therefore, that the solution given above may represent the motion of a sphere in a rotating liquid more closely than the ordinary irrotational solution for a sphere moving in an infinite fluid at rest represents the actual flow in that case.

The surfaces $\psi =$ constant are surfaces of revolution and the stream-lines are spirals wrapped on these surfaces. The sections of the surfaces $\psi =$ constant by an axial plane may be called the stream-lines of the motion in the axial plane. These stream-lines are shown for a particular case in fig. (2). The case chosen is that of a sphere moving along the axis of a

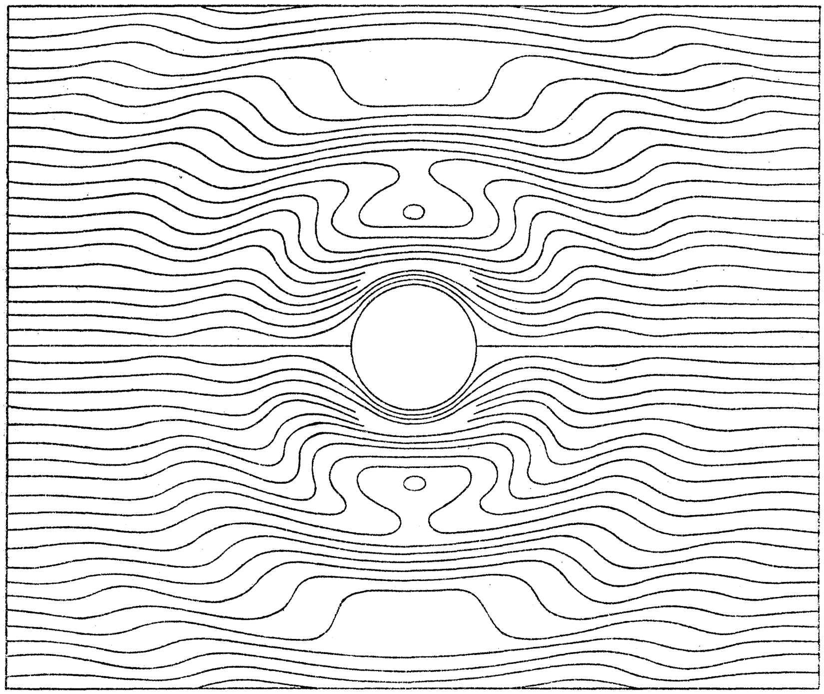

Fig. 2.—Stream-lines due to the motion of a sphere in a rotating fluid. Case when $\mu = 2\pi$.

rotating fluid at such a speed that it travels a distance equal to its diameter (*i.e.*, $\mu = 2\pi$) during each revolution of the liquid, and the particular solution for which there is no slipping between the sphere and the liquid is chosen. It will be seen that the stream-lines are very different from those which surround a sphere moving in a non-rotating liquid.

Propagation of an Isolated Disturbance along the Axis of a Rotating Fluid.

An interesting point about the motion represented by (15) is that it is possible to reduce the radius of the sphere to zero while still retaining a finite velocity in the disturbed motion.

Taking a very small, μ becomes small, and (14) becomes $\tan \epsilon = 0$, so that $\epsilon = 0$, (15) then becomes

$$f = \frac{U}{2k^2}\left[z^2 + 3\left(\cos z - \frac{\sin z}{z}\right)\right]. \tag{16}$$

It can be verified, by substituting in the original equations of motion, that

$$\psi = \frac{U}{2k^2}\left[k^2r^2 + 3\left(\cos kr - \frac{\sin kr}{kr}\right)\right]\sin^2\theta, \tag{17}$$

is a solution of the equations of motion, (17) also represents a motion for which the velocity and the pressure are finite and continuous at the origin. The motion consists of a kind of non-rotating core* of liquid propagated with velocity U along the axis of a rotating liquid. It is analogous to the motion produced by a vortex ring, but in an inverse sense.

The stream-lines, due to the motion of a non-rotating core in a rotating fluid, are shown in figs. 3 and 4. The stream-lines of the steady motion, relative to the moving core, are shown in fig. 3. It will be seen that there are no closed stream-lines, so that the disturbance does not carry any fluid with it.

The stream-lines, relative to the main body of the liquid, are shown in fig. 4. It will be seen that the central part of the disturbance resembles Hill's spherical vortex,† and that it is surrounded by spherical waves which travel with it. The analogy between the present disturbance and a spherical vortex is only superficial, for the vortex ring is a mass of rotating fluid which can move through a non-rotating fluid. The present disturbance is a type which could only be propagated in a rotating fluid, and it consists of a core which rotates more slowly than the surrounding fluid and moves parallel to the axis of rotation.

Wave Systems in a Rotating Fluid.

An essential feature of the motions described above is the system of spherical waves which accompany the moving sphere or moving disturbance. That a system of waves would accompany a body moving in a rotating fluid is to be expected. It has been pointed out by Lord Kelvin‡ that rotation

* Region on the axis where the rotation about the axis is small compared with the rotation of the fluid as a whole.

† M. J. M. Hill, "On a Spherical Vortex," 'Phil. Trans.,' A, 1894.

‡ Kelvin's 'Mathl. and Physical Papers,' vol. 4, pp. 152, 170.

confers on a fluid certain properties resembling those of an elastic solid, and in particular, a rotating fluid can transmit waves.

In order that a system of waves may accompany a disturbance which moves with velocity U along any axis, it is necessary that the velocity of the wave along a normal to the wave-front should be $U\cos\alpha$, where α is the angle between this normal and the direction of the axis along which the disturbance is moving. It is easy to show that a plane wave, of given wave-length, moves in a rotating fluid with velocity proportional to $\cos\alpha$, so that the required condition is satisfied.

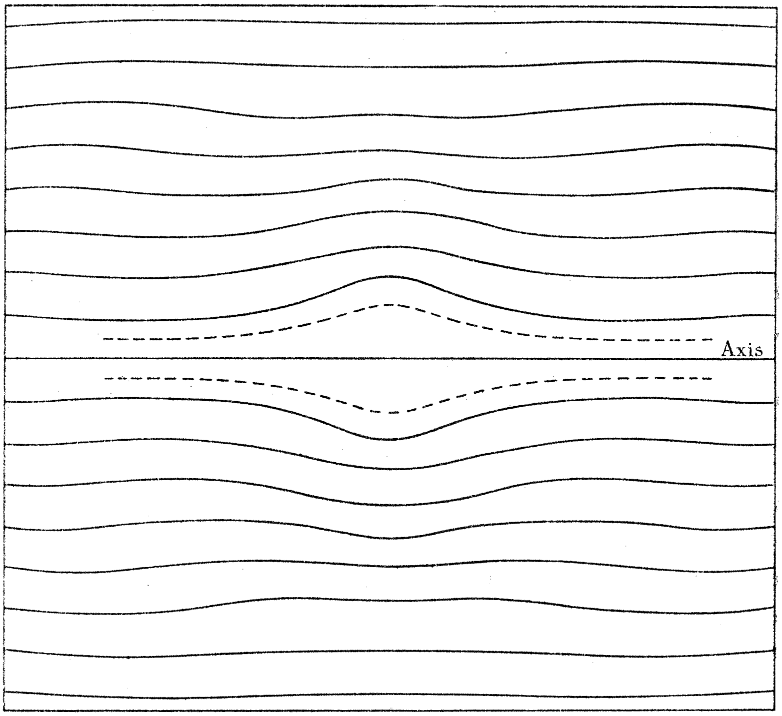

FIG. 3.—Stream-lines relative to the centre of the disturbance due to a non-rotating core travelling along the axis of a rotating fluid.

Writing the equations of motion of a fluid relative to rotating rectangular axes in the form

$$\left.\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-2\Omega u &= -\frac{\partial}{\partial x}\left(\frac{p}{\rho}-\Omega^2 r^2\right)\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+2\Omega u &= -\frac{\partial}{\partial y}\left(\frac{p}{\rho}-\Omega^2 r^2\right)\\ \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} &= -\frac{\partial}{\partial z}\left(\frac{p}{\rho}-\Omega^2 r^2\right)\end{aligned}\right\}, \qquad (18)$$

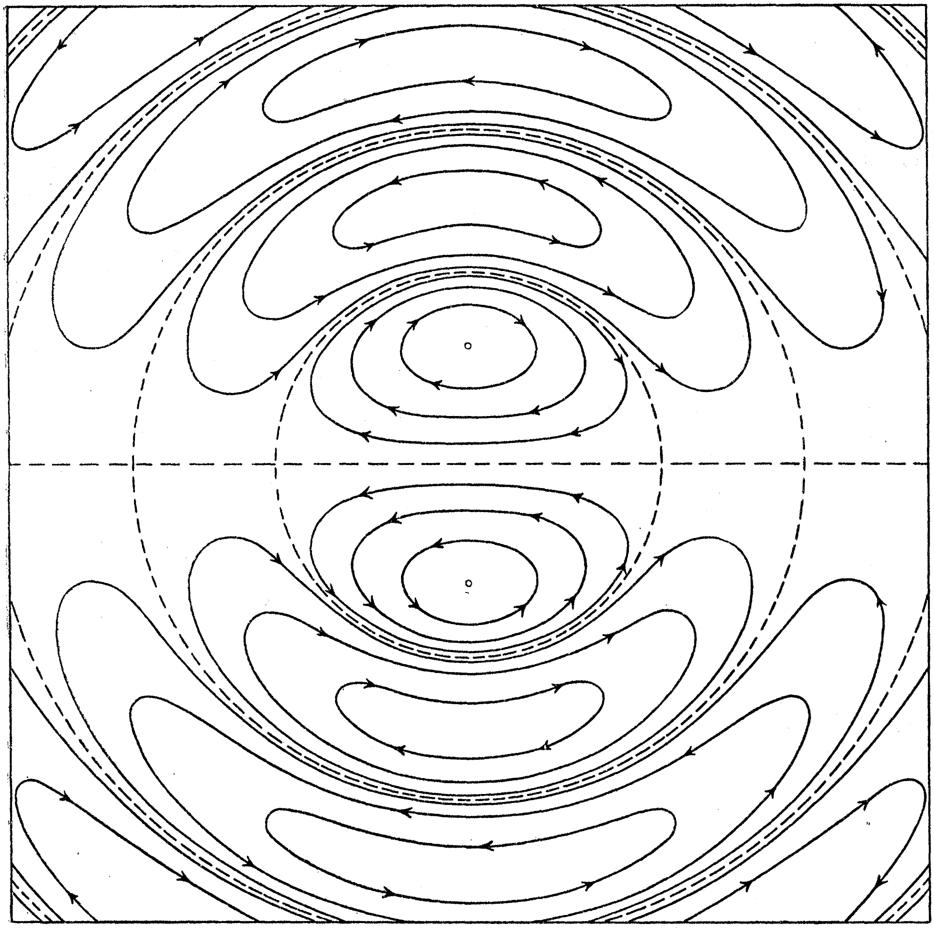

Fig. 4.—Stream-lines relative to the main body of the fluid in the disturbance due to a non-rotating core propagated along the axis of a rotating fluid.

and the equation of continuity as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0,$$

it will be seen at once that these equations are satisfied by

$$(u,\ v,\ w,\ p/\rho-\Omega^2r^2) = (\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{P})\, e^{i(ax+by+cz+nt)}$$

provided*

$$\left.\begin{aligned} ni\mathrm{A}-2\Omega\mathrm{B} &= -ia\mathrm{P} \\ ni\mathrm{B}+2\Omega\mathrm{A} &= -ib\mathrm{P} \\ ni\mathrm{C} \qquad &\doteq ic\mathrm{P} \\ \mathrm{A}a+\mathrm{B}b+\mathrm{C}c &= 0 \end{aligned}\right\}. \tag{19}$$

* The terms involving products of u and v all vanish in this solution on account of the relation $\mathrm{A}a+\mathrm{B}b+\mathrm{C}c = 0$, a condition which implies that the motion of the fluid is confined to the plane of the wave front.

Eliminating A, B, C, P it will be found that

$$a^2+b^2+c^2 = 4\Omega^2c^2/n^2. \tag{20}$$

Hence $$n/2\Omega = c\,(a^2+b^2+c^2)^{-\frac{1}{2}};$$

but $c\,(a^2+b^2+c^2)^{-\frac{1}{2}} = \cos\alpha$, hence

$$n/2\Omega = \cos\alpha. \tag{21}$$

The velocity of the waves represented by (18) is $n\lambda/2\pi$, where λ is the wave-length. Hence, from (21), the velocity is $\Omega\lambda\cos\alpha/\pi$, which is proportional to $\cos\alpha$ if λ is constant. It appears, therefore, that a spherical system of waves of length $U\pi/\Omega$ can be propagated with velocity U parallel to the axis of a rotating fluid.

It is an unusual feature of both the plane and the spherical types of wave that the amplitudes are not limited to small motions.

Experimental Demonstration of the Existence of a Non-rotating Sheath of Fluid Round a Sphere Moving in a Rotating Fluid.

It is difficult to realise a practical demonstration that any of the types of motion, considered above, actually exist in any real fluid; on the other hand, I have been able to show experimentally that, when a sphere moves along the axis of a rotating fluid, it is surrounded by a sheath of fluid which does not rotate with the rest of the fluid. That this result is to be expected is clear from equations (7), for at the surface of the sphere $f = 0$, so that $w = 0$, *i.e.*, the fluid immediately in contact with the sphere has no tendency to rotate it.

The apparatus with which this was demonstrated is shown in fig. 5. In this diagram G is a glass cylinder full of water, A is the axis about which the cylinder is made to rotate uniformly, B is the sphere which consists of a light celluloid ball, of the type used in the game of ping-pong. T is a thread which holds the ball down. The ball can be moved at a uniform speed along the axis by unrolling the thread, T, at a uniform speed from a reel R. This reel is mounted on a spindle fixed in the middle of a lid, C, which is fitted over the glass cylinder G. The thread passed from R through a series of small eyes, E, the last of which was in the centre of the bottom of the cylinder, and the ball, B, was painted with black and white quadrants, so as to make it easy to see whether it was rotating.

The apparatus was first rotated uniformly for some time till the ball and liquid were both rotating as a solid body with the glass cylinder. To perform the experiment, the reel, R, was suddenly fixed so that the thread, T, wound round the reel at a uniform rate. This gave the ball, B, a uniform speed along the axis of the cylinder.

It was found that the ball stopped rotating directly it started moving along the axis. As soon as the reel was released, so that the ball stopped moving along the axis, it quickly picked up the rotation of the rest of the system once more. To perform the experiment successfully, it was found necessary to use the lightest possible type of ball, to use a thin single or plaited silk thread, and to take great care that it was not twisted at the time of the experiment. To ensure success, it was found necessary also to make the ball move at a rate greater than about one diameter per revolution of the system (*i.e.*, $\mu < 2\pi$). If the ball travelled more slowly than this it was found that it did not stop rotating, and investigation of the stream-lines with coloured water, showed that a column of liquid, of the same diameter as the sphere, was apparently pushed along in front of the sphere. This observation suggests that the explanation of the rotation of the sphere, when it moves slowly along the axis, is that the stream-line, $\psi = 0$, does not keep to the surface of the sphere.

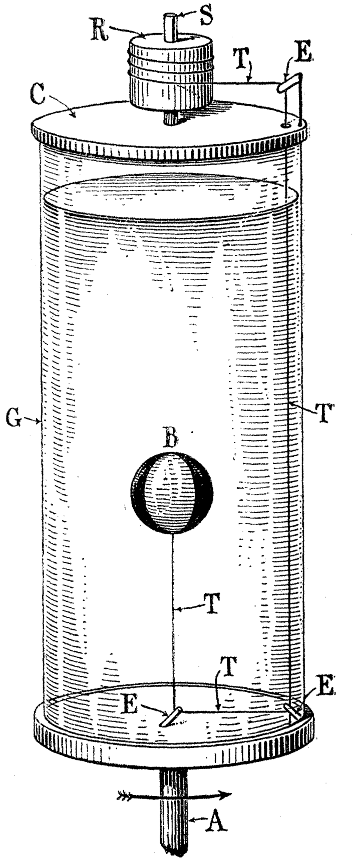

FIG. 5.—Apparatus intended to demonstrate that a sphere is surrounded by a non-rotating sheath of liquid when it travels along the axis of a rotating fluid.

In the course of these experiments, it was noticed that if the sphere was stopped suddenly when half-way up the cylinder, and if there was some colouring matter present to show up the motion, a mass of liquid appeared to detach itself from the sphere, and to continue moving along the axis of rotation with the same velocity as that with which the sphere had been moving. The impression produced on the author's mind was that the flow was similar to that shown in fig. 3. This conclusion, however, should be treated with reserve.