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Particles falling in a rotating fluid

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Particles falling in a fluid rotating about a horizontal axis can go into quasistable orbits. An analytical solution of the particle's motion is derived. This model may be applicable to suspension of diatoms and other particles in the ocean.

INTRODUCTION

Laboratory centrifuges as discussed in a previous paper[1] are always oriented with a vertical axis of rotation. There are many naturally occurring rotating systems with similar orientations, such as tornadoes, maelstroms, dust devils, and the Indianapolis 500. Nature also provides circumstances where the axis of rotation is horizontal. These can occur in clear air turbulence, flow over rippled sand beds, and turbulent eddies in the ocean. In this paper I deal theoretically with the interaction between particles and a fluid rotating about a horizontal axis. This problem is particularly pertinent to the suspension of diatoms and other microscopic plants on the ocean surface, to suspension of sediments and to the suspensions and distribution of pollutants. In a previous paper[2] some experimental aspects of this problem were observed and analyzed.

In the case of diatoms, it is necessary that they maintain themselves near the ocean surface in order to receive sunlight vital for photosynthesis. However, many of them are denser than seawater and tend to sink to the bottom of a bucket of water when scooped from the ocean. Yet these organisms do exist and reproduce in abundance at some locations, so there must be some mechanism that maintains them near the surface. There are upwelling regions in the ocean where the vertical currents are sufficiently strong in the upward direction, not only to maintain diatoms near the surface, but to bring nutrients vital to diatom growth from the deeper realms where the nutrient maxima occur. As diatoms are distributed over regions where upwelling is not present, there must be other mechanisms. Indeed, this analysis shows that particles denser than the surrounding fluid can maintain themselves in a rotating fluid cell for times much longer than would be inferred from their sinking rate.

SIMPLIFIED ANALYSIS

In order to model a particle falling in a rotating cell, I assumed that the fluid was rotating about a horizontal axis as if it were a solid body. Thus the angular velocity is constant at all points within the cell. As demonstrated, both theoretically and experimentally, this condition approximates the velocity field near the center of an eddy.[3] The linear velocity of the fluid within the cell must at some point be greater than the terminal velocity of the sinking particle or it will not be able to achieve a stable or quasistable orbit.

The major forces on the particle are the forces of gravity and of buoyancy and the drag force exerted by the rotating fluid. In all of the cases considered, viscous drag, proportional to the first power of the velocity of the particles relative to the fluid, is a reasonable representation. For example, the Reynolds number of diatoms in the ocean is about 0.01.

For the simplified analysis based on the major forces, the equations in Cartesian coordinates, as shown in Fig. 1, are

$$x \text{ direction:} \quad -\alpha(\dot{x} + r\sin\theta\,\omega) = 0 \qquad (1)$$

$$y \text{ direction:} \quad -\alpha(\dot{y} - r\cos\theta\,\omega) - (\rho - \rho_f)Vg = 0$$

or

$$-\alpha(\dot{x} + y\omega) = 0,$$

$$-\alpha(\dot{y} - x\omega) - m_{\text{eff}}g = 0,$$

where $\alpha = 6\pi\eta a$ assuming Stokes law for viscous flow and $m_{\text{eff}} = (\rho - \rho_f)V$ is the effective mass under influence of gravity with η as the fluid viscosity; a, ρ, and V as the particle's radius, density, and volume, respectively and ρ_f as the fluid density; ω is the fluid's angular velocity and g is the acceleration of gravity. These equations are easily solved:

$$x = K\sin\omega t + m_{\text{eff}}g/\alpha\omega,$$

$$y = K\cos\omega t,$$

where $m_{\text{eff}}g/\alpha = v_T$, the terminal velocity of the particle. The solution is a series of concentric circular orbits centered at $x = +v_T/\omega$ and $y = 0$. The radius of the orbit K depends upon the initial conditions for the particle. This solution also applies for particles less dense than the fluid where $x = -v_T/\omega$ and $y = 0$ would be the center of the orbit. The angular velocity of the orbit is the same as that of the rotating fluid. Thus, according to these solutions, the particles would go into stable circular orbits around a center where their terminal velocity is equal to the upward velocity of the rotating fluid. At first impression, it is rather amazing that particles denser than the fluid can go into stable orbits.

MORE COMPLETE SOLUTION

There are several perturbing forces and departures from the situation as idealized in Eq. (1). In this section I include some of these effects. Figure 2 shows all of the forces on a rotating sphere, assuming that there are no lift forces. In some circumstances there would be a small lift force directed away from the fluid's axis of rotation.[4] However, Bagnold[5] found that there was negligible lift on a rotating or a nonrotating sphere in laminar shear flow provided it was not near a boundary. In order to preserve the simplicity of the analysis, I have neglected lift force as it introduces

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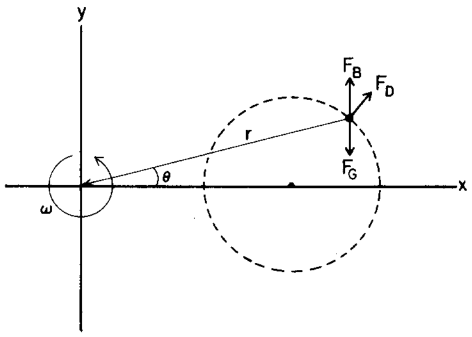

Fig. 1. Major forces on a ball in a rotating fluid. F_B is the buoyancy force, F_G is the gravitational force, and F_D is the drag force. The dotted circle is the ball's orbit.

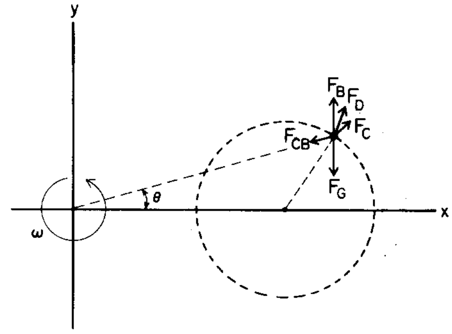

Fig. 2. Perturbation forces on a ball in a rotating fluid. F_C is the centrifugal force on the ball in its own orbit. F_{CB} is the centrifugal buoyancy force. The other forces are defined in Fig. 1. Lengths of the arrows are not necessarily to scale.

nonlinearities destroying the analytical solution. Neglecting lift forces is probably a reasonable assumption for a large two-dimensional vortex with no boundary effects on the particle. In some cases, the magnitude of the lift force may be comparable to that of other perturbing forces considered here.[2] Of course, I have neglected other effects too, such as the inertial forces of the fluid. An exact solution would require solving the Navier-Stokes equation with the proper boundary conditions on the particle. However, the simpler analysis presented here gives a good feeling for the interaction between the particle and the fluid.

The centrifugal "buoyancy" force is due to the pressure gradient created in the fluid by the rotation. This force is directed radially inward and is proportional to the distance from the center of the cylinder.[1] For the case of a sphere this force is

$$F_{\mathrm{CB}} = -\rho_f V \omega^2 \mathbf{r} = -\rho_f V \omega^2 (x\hat{i} + y\hat{j}).$$

Including the inertial terms $m\ddot{x}$ and $m\ddot{y}$ which will provide for the centrifugal force experienced by the particle in its own orbit, the equations of motion are

$$m\ddot{x} = -\alpha(\dot{x} + y\omega) - \rho_f V \omega^2 x,$$

$$m\ddot{y} = -\alpha(\dot{y} - x\omega) - \rho_f V \omega^2 y - V(\rho - \rho_f)g.$$

Both of these new terms are of the order of 10^{-4} of the other terms for small particles in viscous flow. The solution to these equations are derived in the Appendix and take the general form:

$$x = (Ke^{-At} + K'e^{[(1-B)/A]\omega^2 t}) \sin\omega t + v_T/\omega,$$

$$y = (Ke^{-At} + K'e^{[(1-B)/A]\omega^2 t}) \cos\omega t,$$

where $A = \alpha/m$ and $B = \rho_f V/m = \rho_f/\rho$.

Several things are apparent. First, the center of the orbit is still located at $x = v_T/\omega$; $y = 0$. The frequency of orbital rotation is the same as that of the eddy. These conditions also apply for the simple analysis. However, in the more sophisticated analysis there are transient terms. The terms including e^{-At} decrease to zero in very short times and can be neglected. This term corresponds to the time interval required for the particle to reach terminal velocity which is of the order of milliseconds for the range of values pertinent here. The other term is more interesting, having three distinct domains.

For $B = \rho_f/\rho = 1$, the orbit is stable, circular, and centered on the axis of rotation.

For $B > 1$, the orbit decays to its center point and is again stable.

For $B < 1$, the orbit expands indefinitely.

In the case of some small diatoms[6] $B \cong 0.99$ and the orbit should expand with a time constant of several hours. The horizontal eddies probably do not last this long, thus the particles probably move from one eddy to another.

This condition for B can be appreciated more intuitively by considering the perturbing forces on a particle in orbit and balancing them so that the orbit is stable. As seen in Fig. 3, the two perturbing forces are the centrifugal force which depends on the radius of the orbit and the centrifugal buoyancy force which depends upon the distance from the center of the cylinder. The condition of stability is that the sum of the forces at point O is equal in magnitude and oppositely directed to the sum of the forces at O'.

$$V\rho_f \omega^2 d + \rho V \omega^2 a = V\rho_f \omega^2 (d + 2a) - \rho V \omega^2 a$$

which gives $\rho_f/\rho = 1$ for stability. This condition is identical to the one derived from the equations of motion.

CONCLUSION

Based on these equations, a small particle in a less dense, rotating fluid would eventually escape from the rotating cell, or eddy, but with a long time constant. In experiments based on a similar but more complicated analysis, we found that it is possible for particles in a less dense, rotating fluid to go into stable orbits. This behavior can be partly attributed to the experimental setup which was a rotating cylinder filled with viscous fluid (Karo syrup) supporting a falling nylon ball. The effects due to the walls of the cylinder and to the lift forces were responsible for the stable behavior of the

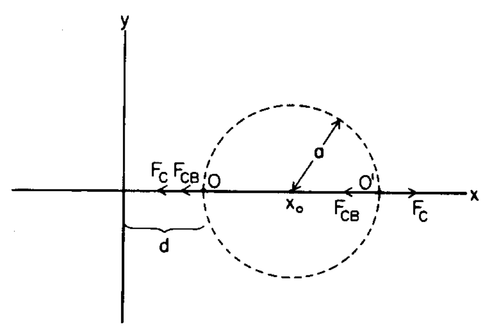

Fig. 3. Perturbing forces on ball along the horizontal direction.

ball. Although it may not be applicable to the ocean, that particular configuration is important for many commercial chemical processes. It is treated in detail in another paper,[2] and may be adapted as a laboratory demonstration of the theory developed in this paper.

The ocean does not have walls; thus studies of the behavior of particles in vertical eddies where the boundary effects are negligible could give valuable insight not only into the behavior of the particles, but also of the turbulence itself.

APPENDIX: SOLUTION OF THE EQUATIONS OF MOTION

$$m\ddot{x} = -\alpha(\dot{x} + y\omega) - \rho_f V\omega^2 x.$$

$$m\ddot{y} = -\alpha(\dot{y} - x\omega) - \rho_f V\omega^2 y - V(\rho - \rho_f)g.$$

Substituting

$$A = \frac{\alpha}{m}; \qquad B = \frac{\rho_f V}{m} = \frac{\rho_f}{\rho}; \qquad C = \frac{V(\rho - \rho_f)g}{m};$$

and using D as the derivative with respect to time, we have

$$D^2x = -A(Dx + y\omega) - B\omega^2 x,$$

$$D^2y = -A(Dy - x\omega) - B\omega^2 y - C.$$

These two second-order differential equations can be reduced to four first-order equations as follows:

$$Dx - u = 0,$$

$$Dy - v = 0,$$

$$Du + Au + A\omega y + B\omega^2 x = 0,$$

$$Dv + Av - A\omega x + B\omega^2 y = -C.$$

The homogeneous solutions to these equations can be obtained by solving the characteristic equation represented by the determinant

$$\begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ B\omega^2 & A\omega & \lambda + A & 0 \\ -A\omega & B\omega^2 & 0 & \lambda + A \end{vmatrix} = 0.$$

Where the λ's are the exponents in solutions of the form

$$x = \sum_j K_j e^{\lambda_j t}.$$

The solutions of the characteristic equation are

$$\lambda = \frac{-A \pm [A^2 - 4(B\omega^2 \pm iA\omega)]^{1/2}}{2}.$$

These solutions can be modified to the more useful form

$$\lambda = \tfrac{1}{2}\left\{-A \pm [(A^2 - 4B\omega^2)^2 + (4A\omega)^2]^{1/4} \times \exp\left[\pm i\tfrac{1}{2}\tan^{-1}\left(\frac{4A\omega}{(A^2 - 4B\omega^2)}\right)\right]\right\}.$$

In all cases of interest where the particle is small and the rotation of the eddy is relatively slow, $B\omega \ll A$.

Separating the real and imaginary parts of the λ's we get

$$\lambda = \frac{1}{2}\left\{-A \pm \left(\frac{r}{2}\right)^{1/2}\left[\left(1 + \frac{A^2 - 4B\omega^2}{r}\right)^{1/2} + i\left(1 - \frac{A^2 - 4B\omega^2}{r}\right)^{1/2}\right]\right\}$$

where $r^2 = (A^2 - 4B\omega^2)^2 + (4A\omega)^2$. Eliminating all of the higher-order terms, these expressions reduce to

$$\lambda_1 = -A \pm i\omega,$$

$$\lambda_2 = [(1 - B)/A]\omega^2 \pm i\omega.$$

The final solutions would be

$$x = e^{-At}(K_1 e^{i\omega t} + K_2 e^{-i\omega t}) + e^{[1-B)/A]\omega^2 t}(K_3 e^{i\omega t} + K_4 e^{-i\omega t}) + \frac{V_T}{\omega},$$

$$y = e^{-At}(K_5 e^{i\omega t} + K_6 e^{-i\omega t}) + e^{[(1-B)/A]\omega^2 t}(K_7 e^{i\omega t} + K_8 e^{-i\omega t}),$$

where the nonhomogeneous solution has been added. The constants may be evaluated from boundary conditions and by substitution in the original equations. With appropriate approximations the solutions reduce to

$$x = (Ke^{-At} + K'e^{[(1-B)/A]\omega^2 t})\sin\omega t + v_T/\omega,$$

$$y = (Ke^{-At} + K'e^{[(1-B)/A]\omega^2 t})\cos\omega t.$$

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