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[1]A. Goldberg, H. M. Schey, and J. L. Schwartz, "Computer-generated motion pictures of one-dimensional quantum-mechanical transmission and reflection phenomena," Am. J. Phys. **35**, 177–186 (1967).
[2]I. Galbraith, Y. S. Ching, and E. Abraham, "Two-dimensional time-dependent quantum-mechanical scattering event," Am. J. Phys. **52**, 60–68 (1984).
[3]S. M. Blinder, "Evolution of a Gaussian wavepacket," Am. J. Phys. **36**, 525–526 (1968).
[4]B. R. Holstein and A. R. Swift, "Spreading wave packets—A cautionary note," Am. J. Phys. **40**, 829 (1972).
[5]J. N. Churchill, "Motion of an electron wave packet in a uniform electric field," Am. J. Phys. **46**, 537–542 (1978).
[6]R. W. Robinett, "Quantum and classical probability distributions for position and momentum," Am. J. Phys. **63**, 823–832 (1995).
[7]M. V. Berry and N. L. Balazs, "Nonspreading wave packets," Am. J. Phys. **47**, 264–267 (1979).
[8]D. M. Greenberger, "Comment on 'Nonspreading wave packets,' " Am. J. Phys. **48**, 256 (1980).
[9]See, e.g., R. L. Liboff, *Introductory Quantum Mechanics* (Addison-Wesley, Reading, MA, 1992), 2nd ed., p. 144.
[10]G. H. Weiss and A. A. Maradudin, "The Baker-Hausdorff formula and a problem in crystal physics," J. Math. Phys. **3**, 771–777 (1962).
[11]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), pp. 63–64.

Stationary waves on cylindrical fluid jets

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An obstacle in the path of a water jet emerging from a tap gives rise to a stationary wave pattern upstream of the obstacle. In this paper, the wavelengths and the damping coefficients of these waves are calculated for various jet radii and velocities. The calculations indicate that the wavelength decreases and the damping coefficient increases with increasing jet velocity, in qualitative agreement with observations. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

The insertion of an obstacle in the path of a fluid jet creates a distinct stationary wave pattern upstream of the obstacle. The wavelength of these waves decreases as the jet velocity is increased and their amplitude, which is greatest at the point of contact, decreases rapidly with distance upstream. These characteristics can be easily observed in a water jet issuing from a faucet, and are referred to in Jearl Walker's collection of problems.[1] A search of the literature revealed few papers[2–5] dealing with stationary waves in cylindrical geometries. These, however, do not explicitly consider the relationship between jet velocity and wavelength nor do they consider viscous effects which, as we will show, are significant. There are a large number of papers dealing with stationary waves in planar geometries. One of the early papers was due to Lord Rayleigh.[6] In that paper, he analyzes the wave pattern created by an obstacle in a flat running stream. That analysis, with minor modifications, has found its way into well known treatises on fluid mechanics.[7] Surface waves on large bodies of water are created by the independent effects of gravity and capillarity (see Ref. 7, p. 461). The resulting waves have very different characteristics from the purely capillary waves considered in this paper.

In this paper, we obtain the wavelength and amplitude damping coefficients for linear capillary waves on cylindrical fluid jets. The "jet stretching" effect of gravity is ignored. Most of the work on waves on cylindrical fluid jets has been driven by questions of jet stability.[8] It was in this context that Lord Rayleigh obtained the dispersion relation for an inviscid fluid jet.[9] In a later paper, he obtained the dispersion for a viscous jet.[10] In the next section, we derive the inviscid dispersion relation and use it to calculate the wavelengths of stationary waves as a function of velocity, for jets of different radii. The derivation is a detailed version of the one presented in Lamb (Ref. 7, p. 472). In the final section, the viscous dispersion relation is derived, closely following Rayleigh's original paper.[10] The dispersion is then used to calculate the damping coefficients of stationary waves.

II. THE INVISCID JET

Consider an infinite, inviscid, cylindrical fluid jet of radius a and velocity V. The fluid density is ρ and the surface tension, T. The axis of the jet is assumed to lie along the z axis. We wish to obtain the dispersion relation for axisymmetric capillary waves that propagate along the surface of the jet. The surface disturbance is denoted by $\eta(z,t)$. The following analysis is carried out in a frame moving with the jet. We assume that the flow that results from the disturbance can be described by a velocity potential. Neglecting the effect of the surrounding medium and assuming incompressibility, the velocity potential satisfies Laplace's equation

$$\phi_{rr}+\frac{1}{r}\,\phi_r+\phi_{zz}=0, \tag{1}$$

in the domain $0\leqslant r\leqslant a+\eta$, subject to

$$\phi_r=\eta_t+\phi_z\eta_z \tag{1a}$$

and

$$\frac{T}{\rho a}-\phi_t-\frac{1}{2}(\phi_z^2+\phi_r^2)=\frac{T}{\rho(1+\eta_z^2)^{1/2}}\left[\frac{1}{r}-\frac{\eta_{zz}}{(1+\eta_z^2)}\right] \tag{1b}$$

at $r=a+\eta$. Equation (1a) states that a particle on the fluid boundary remains on the fluid boundary (kinematic condition). Equation (1b) is obtained by using Bernoulli's equation

$$\frac{p}{\rho}+\phi_t+\tfrac{1}{2}(\phi_z^2+\phi_r^2)=0 \tag{2}$$

and the fact that there is a pressure discontinuity at the surface given by

$$p-p_a=T\left(\frac{1}{R_1}+\frac{1}{R_2}\right), \tag{3}$$

where p_a is the external pressure, p is the pressure just beneath the jet surface, and R_1 and R_2 are the principal radii of curvature of the surface. The constant pressure in the unperturbed jet is chosen as the reference value. Subscripts in the above equations denote derivatives of the subscripted variable with respect to the subscript.

The problem as stated cannot be solved exactly due to the complexity of the boundary conditions (1a) and (1b). In order to proceed, the boundary conditions must be linearized. The linearized boundary conditions are

$$\phi_r=\eta_t, \tag{4a}$$

and

$$\phi_t=\frac{T}{\rho}\left(\frac{\eta}{a^2}+\eta_{zz}\right). \tag{4b}$$

In the linear approximation the boundary conditions are applied at $r=a$ instead of the unknown surface $r=a+\eta$. Solving Eq. (1) subject to the boundary conditions (4a) and (4b) and the condition that the velocity potential be nonsingular at $r=0$ yields

$$\eta(z,t)=\eta_0\cos(kz-\Omega t), \tag{5}$$

and

$$\phi(r,z,t)=\frac{\Omega\eta_0}{kI_1(ka)}I_0(kr)\sin(kz-\Omega t), \tag{6}$$

with the dispersion relation

$$\omega^2=\frac{1}{\beta}\frac{I_1(\alpha)}{I_0(\alpha)}\alpha(\alpha^2-1), \tag{7}$$

where η_0 is the disturbance amplitude, α the dimensionless wave number, ka and ω the dimensionless frequency $\Omega a/V$, and β the Weber number $\rho aV^2/T$. I_0 and I_1 are the modified Bessel functions of the first kind of order 0 and 1, respectively. Note that ω is imaginary for $\alpha<1$. This corresponds to the well-known instability that causes the spontaneous breakup of a fluid jet.[8,9] A physical explanation of the origin of this instability is given in Ref. 11. The stationary waves, however, are amongst the stable waves that occur for $\alpha>1$.

Equation (7) holds in a frame of reference moving along with the jet. The transformation to laboratory coordinates is effected by the mapping $z\mapsto z'-Vt$. The phase in the laboratory frame is therefore, $kz'-(\Omega+kV)t$. Thus, the dispersion in the laboratory frame is obtained by replacing ω by $\omega+\alpha$ in Eq. (7). The equation for the wave number of the stationary waves is then obtained by setting $\omega=0$ in the resulting equation (stationary waves are steady in the laboratory frame). Alternatively, one could simply set $\omega=\alpha$ in Eq. (7) to obtain

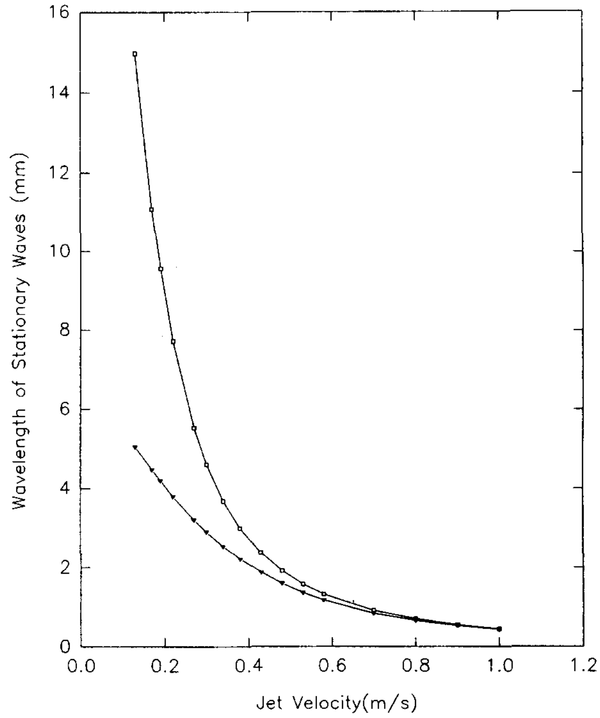

Fig. 1. Wavelength of stationary waves as a function of jet velocity. The triangle denotes the 1-mm radius jet and the unfilled square denotes the 5-mm radius jet.

$$\alpha=\sqrt{\frac{1}{\beta}\frac{I_1(\alpha)}{I_0(\alpha)}\alpha(\alpha^2-1)}. \tag{8}$$

This equation was solved numerically using the Newton–Raphson method. The modified Bessel functions, I_0 and I_1, were evaluated using algorithms adapted from Ref. 12.

In Fig. 1 we show a plot of the wavelength of the stationary waves, λ $(=2\pi a/\alpha)$, as a function of velocity for water jets ($\rho=1000$ kg/m^3, $T=0.073$ N/m) of radius 1 and 5 mm. The wavelength is longest at low velocities and decreases rapidly as the velocity is increased. This is in qualitative agreement with observations made on a $\approx$1-mm radius water jet issuing from a tap. It should be mentioned that the minimum jetting velocity for a 1 mm jet is $\approx$0.15 m/s. Below this velocity the flow is in the form of drops rather than a jet. From the plot, it is also seen that at a given velocity the wavelength is greater for the thicker jet. It is difficult to observe this variation in jets from household taps as the radius cannot be varied without changing the flow rate. The situation is further complicated by the presence of gravity which causes an increase in velocity, and a consequent decrease in radius, along the length of the jet. From the figure, it is also seen that the wavelength becomes independent of the jet radius at high velocities. This can also be seen by taking the limit of Eq. (8) as $k\rightarrow\infty$. In this limit, $I_1(\alpha)/I_0(\alpha)\rightarrow 1$ and we recover the expression for wavelength of capillary waves on a flat surface, $\lambda=2\pi T/\rho V^2$.

It is interesting to note that group velocity of the stationary waves is greater than the jet velocity. This is required if the waves are to be established upstream of the disturbance. It can be shown that the wave energy is propagated at the group velocity, not the phase velocity (see the Appendix). Thus the energy that is input at the obstacle is carried upstream. If the group and phase velocities were equal (i.e., if the wave were nondispersive), we would have shock wave formation with the accumulation of energy at the obstacle. Stationary wave formation is thus possible only if the waves are dispersive.

III. THE VISCOUS JET

The linearized Navier–Stokes equations for an axisymmetric viscous jet are

$$\rho u_t = -p_r + \mu\left[\frac{(ru)_{rr}}{r} - \frac{(ru)_r}{r^2} + u_{zz}\right], \tag{9a}$$

$$\rho v_t = -p_z + \mu\left[v_{rr} + \frac{v_r}{r} + v_{zz}\right], \tag{9b}$$

where u and v are the radial and axial velocities in the jet frame. The equations hold in a frame moving with the jet velocity. The equation of continuity, assuming incompressibility, is

$$u_r + \frac{u}{r} + v_z = 0. \tag{10}$$

The linearized boundary conditions to be satisfied at $r=a$ are

$$-p + 2\mu u_r = \frac{T}{\rho}\left[\frac{\eta}{a^2} + \eta_{zz}\right], \tag{11a}$$

and

$$\mu(u_z + v_r) = 0. \tag{11b}$$

Equation (11a) is the normal stress balance and (11b) is the tangential stress balance.

The symmetry of the problem permits the definition of a stream function, ψ, with $u=\psi_z/r$ and $v=-\psi_r/r$. Equation (10) is then satisfied identically. The elimination of the pressure from Eqs. (9a) and (9b) yields

$$D^2\left(D^2 - \frac{\rho}{\mu}\frac{\partial}{\partial t}\right)\psi = 0, \tag{12}$$

where

$$D^2 = r\,\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}. \tag{13}$$

The boundary conditions are specified by Eqs. (11a) and (11b) and the condition that there be no singularity in ψ at $r=0$. Assuming a solution proportional to $e^{i(kz-\Omega t)}$, it can be shown that

$$\psi(r,z,t) = [A_1 r I_1(kr) + A_2 r I_1(k_1 r)]e^{i(kz-\Omega t)}, \tag{14}$$

where A_1 and A_2 are constants of integration and $k_1 = \sqrt{k^2 - i\Omega\rho/\mu}$. The dispersion relation is obtained by substituting the above solution in the boundary conditions and eliminating the constants A_1 and A_2. The nondimensionalized dispersion relation is

$$2\alpha^2(y^2+\alpha^2)\,\frac{I_1'(\alpha)}{I_0(\alpha)}\left(1 - \frac{2\alpha y}{y^2+\alpha^2}\,\frac{I_1(\alpha)I_1'(y)}{I_1'(\alpha)I_1(y)}\right) + (y^4-\alpha^4) - \frac{\mathfrak{R}^2}{\beta}\,\frac{I_1(\alpha)}{I_0(\alpha)}\,\alpha(1-\alpha^2) = 0, \tag{15}$$

Table I. Calculated values of α, α_r, and α_i for a water jet of 1-mm radius.

v (m/s)	α (Eq. 8)	α_r (Eq. 15)	α_i (Eq. 15)
0.05	1.038	1.038	0.002
0.10	1.148	1.148	0.005
0.20	1.548	1.548	0.012
0.30	2.168	2.167	0.025
1.00	14.34	14.30	0.69
2.00	56.02	55.38	5.39
3.00	126.8	123.6	17.82

where $y = \sqrt{\alpha^2 - i\mathfrak{R}\omega}$, $\mathfrak{R}$ being the Reynolds number, $\rho a v/\mu$.

To obtain the wave number and damping coefficient for stationary waves, we first note that, in the laboratory frame, these waves will be of the form $e^{-ikz'}$, where k is the complex wave number $k_r + ik_i$. Here, we have assumed that the origin is at the obstacle and the the stationary wave is set up along the negative z' axis. Transforming to the jet frame, it is easily seen that the required wave number and damping coefficient is obtained by setting $\mathrm{Re}(\omega)=\alpha_r$ and $\mathrm{Im}(\omega)=\alpha_i$ in Eq. (15) and solving the resulting transcendental equation. Note that the damping is purely *spatial* in the laboratory frame, but is *spatiotemporal* in the jet frame. The equation was solved using the multivariable Newton–Raphson method. Results from the inviscid calculation were used as initial guesses. For $V<1$ m/s, the complex modified Bessel function was evaluated using a well-known series representation.[13] Asymptotic expansions[14] for the Bessel function were used for $V \geqslant 1$ m/s. Some results of the calculations are displayed in Table I.

For high jet velocities ($V \geqslant 2$ m/s), we have $\alpha \gg 1$. In this limit Eq. (15) reduces to the expression for damped linear capillary waves on planar surfaces (see Ref. 7, p. 627):

$$(2\alpha^2 - i\mathfrak{R}\omega)^2 + \frac{\mathfrak{R}^2}{\beta}\,\alpha^3 - 4\alpha^3\sqrt{\alpha^2 - i\mathfrak{R}\omega} = 0. \tag{16}$$

Using this expression, for $V=3$ m/s, we find that $\alpha_r = 123.1$ and $\alpha_i = 17.83$, which is in good agreement with the values obtained by solving Eq. (15).

In order to get a feel for the magnitude of damping, we have plotted the distance in which the amplitude falls to 1% of its original value as a function of the jet velocity in Fig. 2. The y axis scale is logarithmic. It is seen that these waves are strongly damped; the damping increasing dramatically with increasing jet velocity. Further, as in the case of wavelength, it is seen that the damping becomes independent of jet radius at high enough jet velocity. The strong dependence of damping on jet velocity is qualitatively borne out by observations on a jet emerging from a tap. However, as mentioned in the previous section, it is difficult to make a quantitative comparison between experiments and the theory since the theory ignores gravitational effects altogether.

APPENDIX: ENERGY PROPAGATION VELOCITY

Consider a cross section of the jet. The rate at which work is done by the fluid on one side of the cross section on the fluid on the other side consists of two contributions. One is the contribution of the bulk, and the other is the direct contribution due to the surface tension. Thus, in a linear approximation, we may write

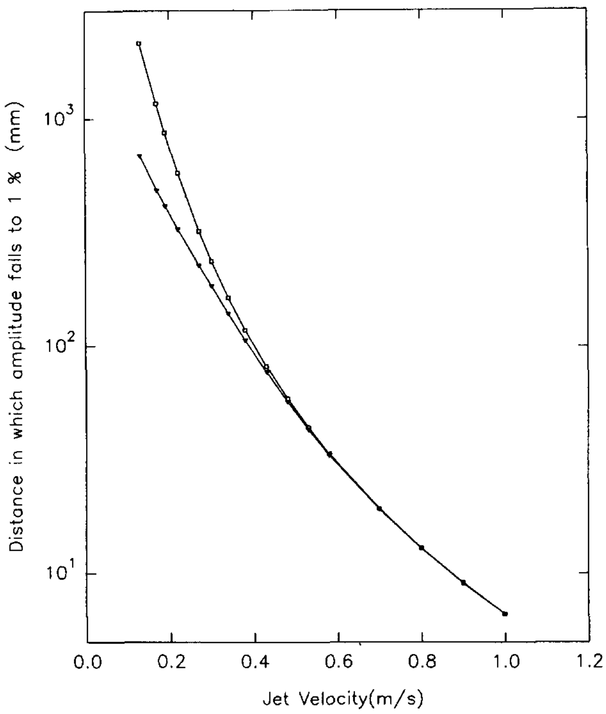

Fig. 2. Distance in which the wave amplitude decreases to 1% of its original value plotted as a function of jet velocity. The triangle denotes the 1-mm radius jet and the unfilled square denotes the 5-mm radius jet.

$$\bar{P}=\overline{\int_0^a p\phi_z 2\pi r\, dr}-\overline{2\pi a T\eta_z\eta_t}, \tag{A1}$$

where P is the rate at which work is done by the fluid on one side on the fluid on the other. The bar indicates that the quantities are averaged over a wavelength. The average energy per unit length of the jet, $\bar{E}$, is given by

$$\bar{E}=2\overline{\int_0^a \frac{\rho}{2}(\phi_z^2+\phi_r^2)2\pi r\, dr}, \tag{A2}$$

where the factor of two is because the average total energy is twice the average kinetic energy in a linear approximation. The velocity at which the energy is propagated, v_e, is then given by

$$v_e=\frac{\bar{P}}{\bar{E}}. \tag{A3}$$

Linearizing Eq. (2), we obtain

$$p=-\rho\phi_t. \tag{A4}$$

$\bar{P}$ and $\bar{E}$ may then be evaluated using Eqs. (5) and (6). The integral

$$\int_0^x I_n^2(\lambda u)du=\frac{-x^2}{2}\left\{I_n'^2(\lambda x)-\left(1+\frac{n^2}{\lambda^2x^2}\right)I_n^2(\lambda x)\right\}, \tag{A5}$$

and the identities

$$2I_n'(x)=I_{n-1}(x)+I_{n+1}(x), \tag{A6}$$

$$\frac{2n}{x}I_n(x)=I_{n-1}(x)-I_{n+1}(x) \tag{A7}$$

are useful in evaluating the expressions. The resulting expressions are

$$\bar{P}=c\bar{E}\left\{\frac{\alpha}{2}\left(\frac{I_0(\alpha)}{I_1(\alpha)}-\frac{I_1(\alpha)}{I_0(\alpha)}\right)+\frac{\alpha^2}{\alpha^2-1}\right\}, \tag{A8}$$

where c is the phase velocity Ω/k. From Eq. (7), it is easily shown that the group velocity, c_g, is given by

$$c_g=\frac{d\Omega}{dk}=c\left\{\frac{\alpha}{2}\left(\frac{I_0(\alpha)}{I_1(\alpha)}-\frac{I_1(\alpha)}{I_0(\alpha)}\right)+\frac{\alpha^2}{\alpha^2-1}\right\}. \tag{A9}$$

Using asymptotic expansions[14] for I_0 and I_1, it can be shown that this expression yields the expected planar surface result, $c_g=3c/2$, in the limit $\alpha\rightarrow\infty$.

Equations (A3), (A8), and (A9) yield the result $v_e=c_g$.

[1]J. Walker, *The Flying Circus of Physics*, (Wiley, New York, 1977), 1st ed., p. 93.
[2]J. Satterly and J. Macpherson, "Stationary Waves in Liquid Jets," Trans. R. Soc. Canada—Series 3 **28**, 177–183 (1934).
[3]J. Satterly and J. C. Strachan, "A Measurement of Surface Tension by Means of Stationary Waves on a Vertical Jet," Trans. R. Soc. Canada—Series 3 **29**, 105–112 (1935).
[4]J. Satterly and J. R. Levitt, "A Study of the Phenomena Associated with a Vertical Jet of Mercury," Trans. R. Soc. Canada—Series 3 **30**, 129–135 (1936).
[5]Y. Wada, "On the Steady Surface Ripples of a Cylindrical Flow," J. Phys. Soc. Jpn. **5**, 259–262 (1950).
[6]Lord Rayleigh, "The Form of Standing Waves on the Surface of a Running Stream," Proc. Lond. Math. Soc. **15**, 69–78 (1883).
[7]Sir H. Lamb, *Hydrodynamics*, (Cambridge U.P., Cambridge, 1932), 6th ed., pp. 468–471.
[8]D. B. Bogy, "Drop Formation in a Circular Liquid Jet," Ann. Rev. Fluid Mech. **11**, 207–228 (1979).
[9]Lord Rayleigh, *The Theory of Sound*, (Dover, New York, 1945), 2nd ed., pp. 351–355.
[10]Lord Rayleigh, "On the Stability of a Cylinder of Viscous Liquid under Capillary Force," Philos. Mag. **34**, 145–154 (1892).
[11]J. S. Trefil, *Introduction to the Physics of Fluids and Solids* (Pergamon, New York, 1975), p. 272.
[12]W. H. Press, B. P. Flannery, S. A. Teukolsky and W. T. Vetterling, *Numerical Recipes* (Cambridge U.P., Cambridge, 1986), 1st ed., pp. 177–178.
[13]G. N. Watson, *A Treatise on the Theory of Bessel Functions*, (Cambridge U.P., Cambridge, 1962), 2nd ed., p. 77.
[14]M. Abramowitz and I. A. Stegun, *A Handbook of Mathematical Functions* (Dover Publications, New York, 1965), pp. 377–378.