The shape function of a free-falling laminar jet: Making use of Bernoulli's equation

Taha Massalha[a)]
Academic Arab College for Education, Haifa 32000, Israel

Rafael M. Digilov[b)]
Department of Education in Technology & Science, Technion–Israel Institute of Technology, Haifa 32000, Israel

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The shape function of a laminar liquid jet issuing from a circular orifice and falling vertically in air under gravity is analyzed. The diameter of the jet is observed to decrease with the axial distance from the nozzle. The governing equation for variation of the jet radius with the axial coordinate is derived from a modified Bernoulli's law, including the interfacial energy density and viscous losses. The analytical solution found in terms of dimensionless group numbers agrees well with experimental data.
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I. INTRODUCTION

When a fluid pours from an outlet into the air, it forms a free-falling stable jet that accelerates, stretches, and narrows under the influence of gravity.[1] The jet flow behavior is of considerable interest in fluid mechanics and engineering practice and has found a wide variety of applications such as the sol-gel process in the production of small fluid particles, the spinning processes in fabrication of polymer fibers, and biomedical devices. Recently, a liquid microjet has been produced[2] that can be used in spacecraft propulsion, fuel injection, mass spectroscopy, and ink-jet printing.

The key challenge when analyzing a jet flow is to find the jet shape function (JSF);[3–8] that is, the relationship between the jet radius r and the axial distance z from the exit orifice. For laminar flow of an isothermal liquid with a density ρ, issuing from a circular orifice of radius R_0 with exit velocity υ_0 in a gravitational field g, dimensional analysis predicts the following functional dependence for the JSF:

$$\tilde{z} = f(\tilde{r}, \mathrm{Fr}, \mathrm{We}, \mathrm{Re}). \tag{1}$$

Here, $\tilde{z} = z/R_0$ and $\tilde{r} = \tilde{z}/R_0$ are the reduced jet length and jet radius, respectively, and the key dimensionless group parameters in the problem are the Froude number (Fr), the Weber number (We) and the Reynolds number (Re), given by

$$\mathrm{Fr} = \frac{\upsilon_0^2}{2R_0 g}, \quad \mathrm{We} = \frac{2R_0\rho\upsilon_0^2}{\gamma}, \quad \mathrm{Re} = \frac{2R_0\rho\upsilon_0}{\eta}. \tag{2}$$

These quantities represent, respectively, the relative effects of gravity (g), surface tension (γ), and viscosity (η) in comparison to inertia, with each defined to be large when inertial effects are comparatively large.

Neglecting the surface tension effect, Clarke[9] derived an analytical JSF for viscous fluids in terms of the Airy function. However, his JSF is valid only for high Re because at low Re the effect of the surface tension becomes more significant than the viscosity[10] and cannot be ignored.[11] Adachi[12] analyzed the effects of the fluid viscosity and surface tension in the asymptotic regions of high and low Reynolds number. No analytical equation for the JSF over a wide range of all three dimensionless group numbers is known. For inviscid fluids (the limit of large Re but still laminar flow), an analytical form of JSF proposed by many authors can be summarized as[13]

$$\tilde{z} = \mathrm{Fr}\left(\frac{1}{\tilde{r}^4} - m\right) - \frac{n}{\mathrm{Bo}}\left(\frac{1}{\tilde{r}} - 1\right), \tag{3}$$

where the first term is due to gravity while the second is the surface tension term due to the curvature of the liquid-air jet surface. Here $\mathrm{Bo} = \mathrm{We}/\mathrm{Fr} = 4R_0^2\rho g/\gamma$ is the Bond number, characterizing the relative effect of gravity with respect to surface tension, while m and n are parameters of the model. According to Kurabayashi,[5] $n = 8$, whereas the slenderness approximation used by Anno[6] yelds $n = 4$. For $n = 0$ and large Bond numbers, Eq. (3) reduces to the well-known Weisbach equation[14]

$$\tilde{z} = \mathrm{Fr}\left(\frac{1}{\tilde{r}^4} - 1\right). \tag{4}$$

The effects of surface tension and viscosity on the form of the stationary jet are active research topics[15–17] and not yet fully understood. In this paper, we develop an analytical approach based on energy considerations to derive the governing differential equation for the jet radius as a function of axial position. We formulate a modified Bernoulli equation[18] for a free-falling jet that includes the jet interfacial energy density and losses due to the fluid viscosity. An analytical equation for the JSF derived in terms of the dimensionless group numbers is compared with experimental observations, and good agreement is obtained.

II. FORMULATION OF THE PROBLEM

Consider isothermal, laminar flow of an incompressible Newtonian fluid with viscosity η, surface tension γ, and density ρ, issuing downward from a circular orifice of radius R_0 into the air with initial velocity υ_0 and falling in a gravitational field $g\hat{\mathbf{z}}$ (z being measured vertically downward) in the form of an axisymmetric jet narrowing downward (see Fig. 1).

For this jet flow, a modifed Bernoulli-type equation[18] along the streamline, including energy losses due to fluid viscosity[19] and free surface energy of the jet, can be written in the form

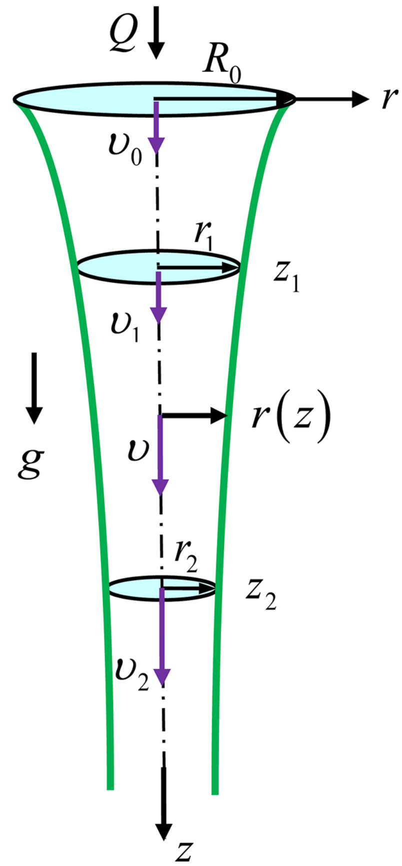


Fig. 1. Sketch of the fluid jet fragment in cylindrical polar coordinates showing relevant variables.

$$P + \frac{\alpha}{2}\rho\upsilon^2 + \rho g z + \gamma\left(\frac{\partial A}{\partial V}\right) + w_{\text{los}} = \text{constant}. \qquad (5)$$

Here P is the mechanical pressure and υ is the velocity averaged over the jet cross section. The coefficient α accounts for the velocity profile distribution:[20] for a uniform profile $\alpha = 1$ while for a nonuniform profile $\alpha > 1$; for laminar flow with a parabolic velocity profile $\alpha = 2$. The term $\rho g z$ is the hydrostatic pressure and $\gamma(\partial A/\partial V)$ represents the jet interfacial energy density.[18] To understand the physical meaning of the latter term, consider a fixed volume element of the liquid dV moving along the streamline through the orifice. The increase in the interfacial surface energy of the jet associated with this volume element is γdA, where dA is the increase in the free surface area of the jet, and $\gamma(dA/dV)$ will be the density of the interfacial surface energy. Finally, the term w_{los} in Eq. (5) denotes the dissipation energy density due to the viscous resistance.

To estimate the surface tension term, notice that the derivative $\partial A/\partial V$ can be written in terms of two components, one being tangential and the other normal to the jet-air interface and can be expressed as

$$\frac{\partial A}{\partial V} = \underbrace{\left(\frac{\partial A}{\partial z}\right)\left(\frac{\partial z}{\partial V}\right)}_{\text{Tangential}} + \underbrace{\left(\frac{\partial A}{\partial r}\right)\left(\frac{\partial r}{\partial V}\right)}_{\text{Normal}}. \qquad (6)$$

For a cylindrical jet segment of volume $dV = \pi r^2 dz$ and free surface $dA = 2\pi r dz$, we find that $(\partial A/\partial z)(\partial z/\partial V) = 2/r$ and $(\partial A/\partial r)(\partial r/\partial V) = 1/r$, where $2/r$ is due to formation of a fluid-air interface and $1/r$ accounts for its curvature. The physical meaning of these two effects can be clarified by considering the corresponding values of the free energies. Whereas $V\gamma(\partial A/\partial z)(\partial z/\partial V) = 2\pi r\gamma dz$ is the interfacial surface energy, the value $V\gamma(\partial A/\partial r)(\partial r/\partial V)$ gives $V\Delta P$ ($\Delta P = \gamma/r$), the excess free energy due to the pressure jump across the interface. These two effects have to be considered together, and therefore the surface tension term in Eq. (5) is

$$\gamma(\partial A/\partial V) = 3\gamma/r. \qquad (7)$$

We apply Eq. (5) for two arbitrarily chosen jet cross sections at points z_1 and z_2, taking into account Eq. (7) and the fact that $P_{1,(2)}$ is the atmospheric pressure, to obtain

$$\frac{\alpha}{2}\rho\upsilon_1^2 - \rho g z_1 + \frac{3\gamma}{r_1} = \frac{\alpha}{2}\rho\upsilon_2^2 - \rho g z_2 + \frac{3\gamma}{r_2} + \Delta w_{\text{los}}, \qquad (8)$$

where $r_{1,(2)}$ is the jet radius at the chosen points 1 and 2, and Δw_{los} is associated with the head pressure loss across the region $l = z_2 - z_1$ due to the viscous resistance.

To calculate Δw_{los}, the Poiseuille equation for viscous flow in a pipe cannot strictly be applied because of the steep change of the velocity profile from a fully developed parabola at the nozzle exit, where $\alpha = 2$, into a "flat" or "plug" profile far away from the nozzle,[21,22] where $\alpha = 1$. However, the Poiseuille equation can be still used to include the viscous losses in the derivation of a suitable mathematical model. To this end, we introduce a correction factor $\delta < 1$ into Poiseuille's equation to get

$$\Delta w_{\text{los}} = \frac{8\delta\eta\upsilon}{r^2}(z_2 - z_1), \qquad (9)$$

where υ and r are the local jet velocity and jet radius. This approach can be justified using dimensional analysis.[23]

Now, understanding that all mathematical derivations refer to a streamline of the laminar jet flow, Eq. (8) can be rewritten as

$$\frac{\alpha}{2}(\upsilon_1^2 - \upsilon_2^2) + g(z_2 - z_1) + \frac{3\gamma}{\rho}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{8\delta\eta}{\rho}\frac{\upsilon}{r^2}(z_2 - z_1), \qquad (10)$$

where α and δ are model parameters that can be determined from experiment.

Taking into account, the uncertainty of the local variables υ and r in the right-hand side of Eq. (10), we replace this equation by its differential analog, setting (see Fig. 1)

$$z_{1,2} = z \mp \Delta z/2; \quad r_{1,2} = r \pm \Delta r/2; \quad \upsilon_{1,2} = \upsilon \mp \Delta\upsilon/2. \qquad (11)$$

After substituting Eq. (11) into Eq. (10), we divide both sides by Δz and take the limit as Δz tends to zero to obtain

$$\alpha\upsilon\frac{d\upsilon}{dz} = g - \frac{3\gamma}{\rho}\frac{1}{r^2}\frac{dr}{dz} - \frac{8\delta\eta}{\rho}\frac{\upsilon}{r^2}. \qquad (12)$$

To simplify the analysis, we use the dimensionless variables, $\tilde{\upsilon} = \upsilon/\upsilon_0$, $\tilde{r} = r/R_0$, and $\tilde{z} = z/R_0$, where $\upsilon_0 = \upsilon(0)$ is the average jet velocity at the exit nozzle and $R_0 = r(0)$ is the exit nozzle radius. As a result, Eq. (12) becomes

$$\alpha\tilde{\upsilon}\frac{d\tilde{\upsilon}}{d\tilde{z}} = \frac{1}{2\mathrm{Fr}} - \frac{1}{\mathrm{We}}\frac{6}{\tilde{r}^2}\frac{d\tilde{r}}{d\tilde{z}} - \frac{16\delta}{\mathrm{Re}}\frac{\tilde{\upsilon}}{\tilde{r}^2}, \tag{13}$$

where the dimensionless group numbers, Fr, We, and Re are as in Eq. (2), and $\tilde{r}$ and $\tilde{\upsilon}$ are related by the non-dimensional equation of continuity,

$$\tilde{r}^2(\tilde{z})\tilde{\upsilon}(\tilde{z}) = 1. \tag{14}$$

Differentiating Eq. (14) with respect to $\tilde{z}$, we find

$$-\frac{2}{\tilde{r}^2}\frac{d\tilde{r}}{d\tilde{z}} = \frac{1}{\tilde{\upsilon}^{1/2}(\tilde{z})}\frac{d\tilde{\upsilon}}{d\tilde{z}}, \tag{15}$$

and then using Eqs. (14) and (15) in Eq. (13), we obtain

$$\alpha\tilde{\upsilon}\frac{d\tilde{\upsilon}}{d\tilde{z}} = \frac{1}{2\mathrm{Fr}} + \frac{1}{\mathrm{We}}\frac{3}{\tilde{\upsilon}^{1/2}}\frac{d\tilde{\upsilon}}{d\tilde{z}} - \frac{16\delta}{\mathrm{Re}}\tilde{\upsilon}^2. \tag{16}$$

Equation (16) implies an increase in viscous resistance with the average jet flow velocity in proportion to $\tilde{\upsilon}^2$. This means that at a sufficient distance from orifice, tangential acceleration along the streamline vanishes, i.e., $\tilde{\upsilon}(d\tilde{\upsilon}/d\tilde{z})|_{z\to\infty} = 0$, resulting in asymptotic scaling for the jet flow velocity and jet radius far away from the nozzle:

$$\tilde{\upsilon}_\infty = (\Lambda/32\delta)^{1/2}; \quad \tilde{r}_\infty = (32\delta/\Lambda)^{1/4}. \tag{17}$$

Here, $\Lambda = \mathrm{Re}/\mathrm{Fr}$ is the renormalized dimensionless group number, describing the ratio of the gravitational force $2R_0\rho g$ to the viscous resistance force $\eta\upsilon_0/2R_0$.

In the same manner, we obtain from Eq. (13) the governing equation for the dimensionless jet radius $\tilde{r}(\tilde{z})$:

$$-2\alpha\frac{1}{\tilde{r}^5}\frac{d\tilde{r}}{d\tilde{z}} = \frac{1}{2\mathrm{Fr}} - \frac{6}{\mathrm{We}}\frac{1}{\tilde{r}^2}\frac{d\tilde{r}}{d\tilde{z}} - \frac{16\delta}{\mathrm{Re}}\frac{1}{\tilde{r}^4}, \tag{18}$$

which can be directly integrated with the initial condition $\tilde{r}(0) = 1$ to yield

$$\tilde{z} = \frac{3}{\tilde{r}_\infty \mathrm{Bo}}\left[2\arctan(\tilde{r}_\infty/\tilde{r}) - 2\arctan(\tilde{r}_\infty) - \ln\left(\frac{1-\tilde{r}_\infty/\tilde{r}}{1+\tilde{r}_\infty/\tilde{r}}\right)\left(\frac{1+\tilde{r}_\infty}{1-\tilde{r}_\infty}\right)\right] - \frac{\alpha\mathrm{Fr}}{\tilde{r}_\infty^4}\ln\frac{1-(\tilde{r}_\infty/\tilde{r})^4}{1-\tilde{r}_\infty^4}. \tag{19}$$

The advantage of this analytical JSF is its mathematical consistency modified by the dimensionless group numbers based on the gravitational force. In the physically interesting case when the viscosity relative to gravity becomes negligible ($\Lambda \gg 32\delta$), we can expand all terms in Eq. (19) with $\tilde{r}_\infty < 1$, using the approximations $\arctan(\tilde{r}_\infty/\tilde{r}) \approx \tilde{r}_\infty/\tilde{r}$, $\arctan(\tilde{r}_\infty) \approx \tilde{r}_\infty$, $\ln(1 \pm (\tilde{r}_\infty/\tilde{r})^4) \approx \pm(\tilde{r}_\infty/\tilde{r})^4$, and $\ln(1 \pm \tilde{r}_\infty^4) \approx \pm\tilde{r}_\infty^4$, to obtain

$$\tilde{z} = \alpha\mathrm{Fr}\left(\frac{1}{\tilde{r}^4} - 1\right) + \frac{12}{\mathrm{Bo}}\left(\frac{1}{\tilde{r}} - 1\right), \tag{20}$$

which is Eq. (3) with $n = 12$ and $m = 1$ modified by the factor α. At large Bo, when the surface tension effect becomes negligible compared to gravity, the first term in Eq. (19) can be omitted, resulting in a JSF governed by gravity and viscous resistance:

$$\tilde{z} = -\frac{\alpha\mathrm{Fr}}{\tilde{r}_\infty^4}\ln\frac{1-(\tilde{r}_\infty/\tilde{r})^4}{1-\tilde{r}_\infty^4}, \tag{21}$$

which at $\tilde{r}_\infty < 1$, i.e., $\Lambda \gg 32\delta$, reduces to the Weisbach equation (3) modified by a multiplicative constant α:

$$\tilde{z} = \alpha\mathrm{Fr}\left(\frac{1}{\tilde{r}^4} - 1\right). \tag{22}$$

III. EXPERIMENT

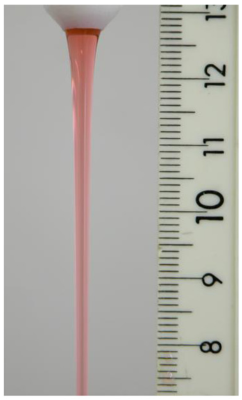
Fluid gravity-fed from a tank was discharged from a nozzle of radius R_0 into a beaker mounted on a force sensor. The flow rate Q was controlled by a valve mounted between the tank and the nozzle. The force sensor, connected through a data logger to a computer, continuously recorded the fluid weight $m(t)g$ exiting from the nozzle as a function of time, so that the average flow rate Q could be determined from the slope of

$$Q = \frac{1}{\rho}\frac{dm(t)}{dt}, \tag{23}$$

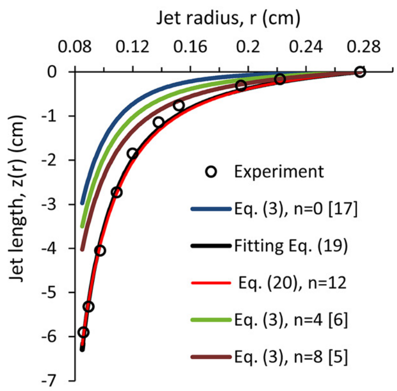
and the average jet velocity at the nozzle exit was calculated as $\upsilon_0 = Q/\pi R_0^2$. The Teflon nozzle of radius $R_0 = 2.775\,\mathrm{mm}$ was cut off sharply and conically at the exit end to prevent the liquid from wetting and attaching to the horizontal end plane facing downward. Thus, the fluid issued from the vertical nozzle was observed to separate from the solid wall at the position of $z = 0$ and $r = R_0$ over the flow rate range typically used in experiments, 4 to 20 cm^3/s. The jet radius r as a function of length z was measured from digital images using ImagePro4 software. Using the nonlinear fitting procedure available in MATHEMATICA,[24] Eq. (19) was fit to measured data of $z(r)$ taking α and δ to be fitting parameters.

IV. RESULTS AND DISCUSSION

Ordinary tap water at temperature 25°C with density $\rho \approx 994.5\,\mathrm{kg\,m^{-3}}$, viscosity $\eta \approx 0.89\,\mathrm{mPa\,s}$, and surface tension $\gamma \approx 0.063\,\mathrm{N\,m^{-1}}$, was tinted to produce a visible and stable laminar jet free-falling under gravity into air (see the photograph in Fig. 2(a), taken at a distance of 40 cm). A millimeter ruler was placed in the plane of the jet for calibrating the distance in the image. At a constant volumetric flow rate $Q \approx 6.97\,\mathrm{cm^3/s}$, the average jet velocity at the nozzle exit was $\upsilon_0 \approx 7.2\,\mathrm{cm/s}$. The dimensionless group numbers computed from Eq. (2) are $\mathrm{Re} = 447$, $\mathrm{Fr} = 0.095$, $\mathrm{Bo} = 4.77$, and $\Lambda = \mathrm{Re}/\mathrm{Fr} = 4700$. The fit of Eq. (19) to experimental data $z(r)$ with $\alpha = 1.397$ and $\delta = 0.2514$ in Fig. 2(b) is excellent. Fit parameters indicate that the velocity profile of the jet flow is not uniform ($\alpha > 1$) and the friction coefficient is $\lambda(\upsilon) = 64\delta/\mathrm{Re} \approx 16/\mathrm{Re}$. The jet radius and the jet velocity far from the nozzle exit calculated by Eq. (17) are, respectively, $r_\infty \approx 0.56\,\mathrm{mm}$ and $\upsilon_\infty \approx 2.4\,\mathrm{m/s}$. As $\Lambda \gg 32\delta \approx 8$ and $r_\infty/R_0 \approx 0.2 < 1$, the viscous losses are negligible and the JSF can be described by Eq. (20), governed by gravity and surface tension forces alone. Indeed, as seen from Fig. 2(b), the fits of Eq. (19) to measured data and predicted by Eq. (20) are identical. For comparison with experimental data and the predictions of Eq. (3), we also show in Fig. 2(b) the JSF for different values of the parameter n. It is seen that Eq. (3) with $n = 0$,[17] $n = 4$,[6] and $n = 8$,[5] only qualitatively describes the experiment data, while the fit



(a)



(b)

Fig. 2. (a) Digital image of vertically falling jet of tinted water issuing from the nozzle of radius $R_0 = 0.2775\,\mathrm{cm}$ with velocity $v_0 \approx 7.2\,\mathrm{cm/s}$; (b) fit of Eq. (19) to experimental data $z(r)$ in comparison with the predictions of Eqs. (3) and (20).

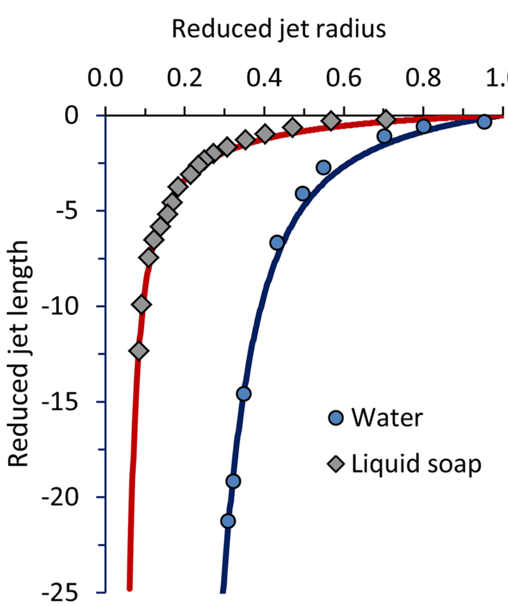


Fig. 3. The surface tension effect on the jet contraction: comparison of liquid soap jet with $\gamma = 35\,\mathrm{mN \cdot m^{-1}}$ and water jet with $\gamma = 70\,\mathrm{mN \cdot m^{-1}}$. Symbols denote experimental data. Solid lines are predicted by Eq. (20).

predicted by Eq. (20) with $n = 12$ is excellent. Moreover, the comparison of the JSF at different n suggests that the surface tension acts against the contraction of the jet. To test the validity of this assumption, we compare in Fig. 3 two experimental JSFs produced by liquid soap with $\gamma \approx 35\,\mathrm{mN \cdot m^{-1}}$ and water with $\gamma \approx 70\,\mathrm{mN \cdot m^{-1}}$. As expected, under the same conditions, the water jet is narrowed less than the liquid soap jet.

V. CONCLUSION

We have considered the effect of surface tension and viscosity on the jet shape function (JSF) of an axially-symmetric, laminar jet of an incompressible Newtonian viscous fluid issuing from a circular orifice and falling vertically under gravity into the atmosphere.

Our main results can be summarized as follows:

1. We have developed a simple analytical model to infer qualitatively and quantitatively the JSF over a wide range of the key dimensionless group numbers.
2. The derived analytical JSF is in good agreement with the observed shape of the free surface of a laminar jet.
3. At high Bond numbers, when surface tension is negligible and in the limit of large Reynolds numbers, when the influence of the fluid viscosity becomes negligible, the JSF reduces in to the modified equations known from the literature.
4. An asymptotic scaling for the jet flow velocity and jet radius far from the exit orifice has been found in terms of the reduced dimensionless group number $\Lambda = \mathrm{Re}/\mathrm{Fr}$, representing the ratio of the gravitational to the viscous resistance force.

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[a]Electronic mail: taham@macam.ac.il
[b]Electronic mail: edurafi@technion.ac.il
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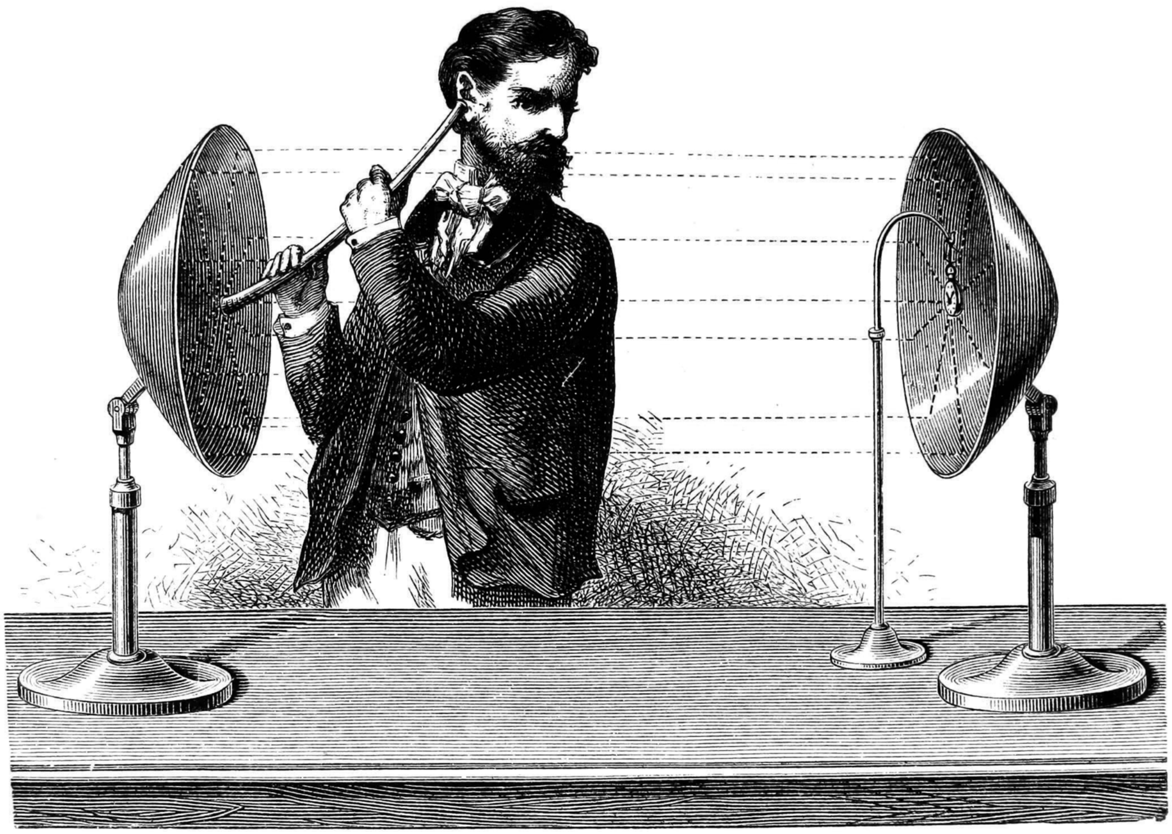

Parabolic Acoustic Mirrors

Sound from a ticking watch placed at the focal point of a parabolic mirror can be picked up by an acoustic wave guide placed at the focal point of a similar, facing mirror. From Amédée Guillemin, *The Forces of Nature*, translated from the French by Mrs. Normal Lockyer and edited by J. Norman Lockyer (Macmillan and Co., London, 1877) pg 142. (Text by Thomas B. Greenslade, Jr., Kenyon College)