# Non-Equilibrium Systems and Irreversible Processes

Adventures in Applied Topology

Vol. 2

# Falaco Solitons, Cosmology, and the Arrow of Time

from a Perspective of Continuous Topological Evolution.

**R. M. Kiehn**
Emeritus Professor of Physics
University of Houston
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## 0.1 Preface

This monograph is the second volume of a series in which topological methods are applied to the study of diverse Non Equilibrium Systems and Irreversible Processes. The three topics considered in this monograph are "Falaco Solitons, Cosmology and the Arrow of Time". These topics, although seemingly unrelated in terms of geometric properties of size, shape and continuous geometric dynamics, appear to have an extraordinary universal equivalence from the perspective of Continuous Topological Evolution (See Vol 1 of this Series, "Non-equilibrium Thermodynamics"). Non-equilibrium systems undergoing irreversible processes most often consist of a collection of diverse, but synergetic and topologically coherent, components of more than one species. The components can consist of atoms, or they may consist of galaxies. The topology of interest in this monograph does not depend upon geometric properties such as size or shape. If the number of components change then topological evolution has taken place. Condensation or merging together is one form of topological evolution, where the number of components changes; such evolutionary changes can be described by continuous processes. These dynamical systems often "self organize" by irreversibly evolving to collective long-lived states, far from equilibrium. They then sustain themselves by "feeding" and delivering "waste" to their environments, but (apparently) will ultimately decay to an equilibrium state of inactivity or death. From a topological point of view, these non-equilibrium thermodynamic systems have an underlying (topological, not geometrical in the sense of size and shape) dynamical theory that makes them appear to be universally equivalent.

Classic equilibrium thermodynamics utilizes statistical methods influenced by the predictable and observable properties of continuous geometric evolution. Historically, the theory of continuous geometric evolution can be used to describe the evolutionary dynamics of particles and fluids. There exist categories of continuous geometric processes of translation and rotation whereby geometric invariant properties of size and shape can be used to encode the "particles". There also exist other categories of continuous geometric processes of deformation (which do not preserve size and shape) but where by topological (deformation) invariant properties can be used to encode the "fluid". Continuous geometric evolution implies that the process can be described by a diffeomorphism (a C1 differentiable map, with a C1 inverse), a constraint which permits the deduction of a unique final state *neighborhood* from given initial data.

For Non Equilibrium Systems and Irreversible Processes, the concept of continuous geometric evolution must be replaced by the concept of continuous topological evolution (See chapter 5 of Vol. 1). Topological change is a necessary feature of continuous irreversible processes. Irreversible processes can not be described by diffeomorphisms. Hence, the concept of tensor fields (which are defined with respect to diffeomorphisms) must be replaced by other mathematical objects which are functionally well behaved with respect to processes which are not diffeomorphisms and which can be used to describe topological change. In this series of monographs the objects which are used to encode the physical system are taken to be "exterior differential forms" as developed and exploited by E. Cartan. Exterior differential forms behave as scalars (or scalar densities) with respect to tensor diffeomorphisms, so they work well when the domain of interest is restricted to the equivalence class of diffeomorphisms and geometrical evolution. More importantly, exterior differential forms are well behaved, in a functional neighborhood sense, with respect to C1 mappings that are *not* homeomorphic. Recall that most tensor fields are not well behaved in a predictive functional *neighborhood* sense, relative to non-homeomorphic maps. In this monograph it is demonstrated that fundamental thermodynamic principles can be extended to describe Non Equilibrium Systems and Irreversible Processes - when physical systems are encoded in terms of exterior differential forms, and subjected to continuous topological, not geometrical, evolution.

The historical use of a geometric diffeomorphic approach (tensor analysis), with emphasis on uniqueness, symmetries and conservation laws, to solve problems in physics has heretofore constrained, if not eliminated, the stated objective of understanding Non Equilibrium Systems and Irreversible Processes. However, geometric methods, borrowing the words of Eugene Wigner, have been "unreasonably effective" in understanding physical phenomena - at least for phenomena that can be approximated by isolated-equilibrium systems and statistical averages. The geometric methods developed historically (and based upon geometry) are time reversal invariant. Although the geometric dimension of such "isolated" systems can be much larger than 2, Caratheodory has demonstrated that the Pfaff Topological dimension is not greater than 2, However, non equilibrium thermodynamic systems undergoing irreversible continuous processes require that the Pfaff Topological dimension must be greater than 2. The topology of the initial state and the topology of the final state are not the same if the process is irreversible.

Paraphrasing Eddington, and due to the insistence of predictive uniqueness (Pfaff topological dimension equal to 2 or less):

*The concepts of aging and the arrow of time have slipped through the net of geometric analysis.*

Most of the references to my earlier publications have been compiled for convenience in Vol 7 "Selected Publications", which is available in paper back form, or in PDF file download format. See www.cartan.pair.com.

## 0.2 Points of Departure

From the outset, it is assumed that the presence of a physical system induces a topology on a set of base variables (say space time). The dynamics of the system refines the topology. This idea is similar to, but different from, the assumption that the presence of matter establishes a metric on a set of base variables.

In this monograph certain physical systems and processes will be studied in terms of a top down topological method, rather than a bottom up method. That is, the physical system will be presumed to have started as a non-equilibrium fluidic system in a turbulent state and subjected to irreversible processes. The Pfaff Topological dimension (See Vol. 1) of the initial state for such systems must be an even number equal to 4 or more. The turbulent system will irreversibly decay to produce topological defects, which are collective, observable, long lived states far from equilibrium (often with an odd Pfaff Topological dimension of 3 or more). In the sense of the Cartan topology, such long lived objects are represented by closed exterior differential forms, which are deformable integral invariants (hence topological properties independent from geometrical properties of size and shape. As the exterior derivative of such closed objects vanishes, they do not have limit points with respect to the Cartan topology. This method is the opposite of the bottom up technique, which assumes the system is in equilibrium (Pfaff Topological dimension of 2 or less), and then examines the possibility that observables are generated by perturbations of the equilibrium state to create defect structures.

The major difference is that the bottom up method starts with a connected topology (Pfaff Topological dimension of 2 or less), while the top down method starts with a disconnected topology (Pfaff Topological dimension of 4 or more). It is possible by continuous maps to evolve from a disconnected topology to a connected topology, but it is impossible to evolve from a connected topology to a disconnected topology in a continuous manner. It is here, via the axiom of topological continuity, where the arrow of time becomes well defined. From a cosmological point of view, the universe will be presumed to be a dilute non-equilibrium turbulent gas (deformably equivalent to of the van der Waals gas and of Pfaff Topological dimension 4) near its critical point. Hence large fluctuations in density are to be expected. These fluctuations in density are presumed to be stars and galaxies that cause the night sky to be inhomogeneous. Certain universal classes of topological defects of odd Pfaff topological dimension will be investigated. One universal class of such objects, Falaco Solitons, can be easily created in a swimming pool. The Falaco Solitons are topologically coherent, but deformable structures, which appear to self organize themselves during thermodynamically irreversible processes of topological change into long lived states far from equilibrium.

As developed in Vol 1, the Cartan topology for such non-equilibrium systems, of Pfaff topological dimension greater than 2, is a disconnected topology, which can support many components (mixed phases). Another way of describing such a topologically disconnected system (of topological defects) is that if solutions exist, there may be more than one solution (non uniqueness) at any geometric point, leading to the notion of envelopes, Huygen wavelets, tangential discontinuities, and edges of regression representing stability limits and the possibility of thermodynamic phase change. Indeed, an important topological property is the number of disconnected parts, which in the treatment of non-equilibrium thermodynamics will be related to the mole number n.

## 0.3 Results

It is remarkable that by using a topological perspective and the axioms for continuous processes (given in detail in Vol 1. and summarized in the following chapters) non-equilibrium systems and irreversible processes can be studied without the use of probability or statistical methods, and without the use of geometric metric constraints and linear connections. The topological method, constructed on a Cartan system of exterior differential forms which are inherently anti-symmetric, emphasizes the anti-symmetric properties of a physical system, where the more geometric and statistical methods, based upon quadratic metric forms and symmetric averages, tend to obscure the anti-symmetry properties.

It is further remarkable that the Jacobian matrix of the coefficients of the 1-form of Action - for those non equilibrium turbulent physical systems of Pfaff topological dimension 4 - leads to a universal thermodynamic phase function represented by a polynomial equation of 4th degree. The universality is related to the singularity theory of non degenerate systems which are equivalent under (small) deformations. The Phase function is constructed in terms of the symmetric similarity invariants of the Jacobian matrix of the component functions that encode the 1-form of Action, $A$. The envelope of the universal Phase function is deformably equivalent to a van der Waals gas. This universal resultant Phase function brings attention to thermodynamic phases that have equivalent (symmetry) structures other than those depending upon size and shape. In general, the exterior differential form method focuses attention on thermodynamic phases that have equivalent deformable topological structures (equivalent Pfaff topological dimension), and which are the result of continuous topological evolution.

Indeed, this resultant universal fourth order Phase function result matches the concepts of Landau $\Psi^4$ mean field theory and phase transitions on one hand , and on the other hand makes contact with the non equilibrium expansion of the universe described by "inflation", and dark matter and dark energy concepts due to a "Higgs" quartic potential below the critical point of a deformable van der Waals gas. The concepts of surface tension (or string theory) can be related to the mean curvature (induced by the molar density) of the universal phase surface, the concepts of temperature and entropy are related to the quadratic or Gauss curvature (induced by the molar density), while the concepts of pressure (of either sign) and interactions are related to the cubic curvatures (induced by the molar density). The theory as presented herein is far from being complete, yet the methods offer a new perspective for analyzing thermodynamic problems. Moreover, the techniques appear to solve the problem of making a marriage between mechanical dynamics and thermodynamics; the methods can be quite useful in the design of new applications previous excluded by assumptions of equilibrium and uniqueness.

The historical limitations of geometric (metric-size-and-shape) and topological (deformation) invariance usually imposed upon theoretical descriptions of nature (especially in relativity theories) are abandoned herein in favor of studying those properties that are homeomorphic invariants, and yet permit description of topological, as well as geometric, change relative to continuous transformations. The methods which are presented herein are based upon Cartan's calculus of exterior differential forms [64], [35]. Exterior differential forms are objects, which, in contrast to tensors, are well behaved with respect to differentiable (continuous) mappings that do not have an inverse (and therefor do not preserve topological properties), and are also well behaved with respect to diffeomorphisms, which are differentiable invertible continuous mappings (and which preserve topological properties). Evolutionary processes will be defined in terms of the action of the Lie differential with respect to vector direction fields acting on differential forms [133]. The Lie differential acting on differential forms is not confined by the diffeomorphic constraints of tensor analysis, and can treat problems of topological change. The method goes beyond the more standard "extremal" techniques based upon the calculus of variations. In most of that which follows, the functions used to define the physical systems will be assumed to be C2 differentiable. The functions that describe processes most often will be assumed to be C2 differentiable as well, but certain C1 processes (inducing tangential discontinuities and wakes) and C0 processes(inducing shocks and first order phase transitions) are of physical interest.

A fundamental result of non equilibrium thermodynamics can be expressed by the statement:

> *Topological change is a necessary condition for a continuous thermodynamic process to be irreversible. .*

Irreversible processes, related to the arrow of time and the biological aging process, require topological evolution and topological change. Current physical theories that describe evolutionary processes (for example, Hamiltonian or Unitary dynamics) usually are formulated in terms of homeomorphisms that emphasize geometrical properties, but do not permit topological change. Hence all such homeomorphic continuous processes are thermodynamically reversible and are inappropriate for the study of continuous topological evolution.

## 0.4 Monograph Site Map

The monograph starts with an experimental observation that highly motivated and sustained the author's research interest in Non Equilibrium Systems and Irreversible Processes. The experiment is easily performed (and has won prizes at state fairs for science projects conducted by high school students in the USA). Chapter 1 goes directly to a discussion of the extraordinary topological defects (known as Falaco Solitons) that can be (and have been) created and studied in a swimming pool. The ability to create Falaco Solitons gives a high level of credence to the fundamental theory of continuous topological evolution. These Falaco Solitons turn out to be locally unstable, but globally stabilized, long lived objects, that are far from equilibrium. Both the experiment and the theory are developed in Chapter 1, where it becomes evident that the Falaco Solitons appear universally among the dynamical system governed by equations of the Navier-Stokes type.

Several challenges were thrown to the "String Theorists" to solve the problem using their methods. There were no replies. Yet it appears now that the solutions given in Chapter 1, without use of String Theory, seem to give an understanding of the problem in terms of non equilibrium thermodynamics.

Chapter 2 begins with a top down model for the universe. The initial motivation came from an argument presented by Landau, in terms of correlations of fluctuations. Herein, the statistical method is overwritten in terms of a cosmology that presumes the universe is deformably equivalent to a non equilibrium van der Waals gas near its critical point. Most of the mathematical development is detailed in chapter 2. Certain mathematical terms and a few useful theorems may be new to some readers. They are introduced without apology or tutorial description, but are sufficiently detailed in Vol 1. For those who need to be brought up to speed with Cartan's concepts of exterior differential forms, a number of textbooks are available [64], [12], [124], [6].

Chapter 3 describes how the concept of (topological) continuity can be used to formulate what has been called the arrow of time. It is demonstrated that homeomorphic physical theories, with evolutionary results that preserve topology, can not describe the details of the arrow of time. No such orientational structure exists for Hamiltonian systems. Recall that non-diffeomorphic maps cannot be used predict *functional forms* for neighborhood of tensor fields. However exterior differential forms are functionally well defined with respect to continuous, but non-homeomorphic maps, in a retrodictive manner. There is a logical difference in continuously evolutionary processes. When topology changes continuously, tensor fields and exterior differential forms are not uniquely predictable in a functional neighborhood sense. However, differential forms are retrodictable in a functional neighborhood sense with respect to such non-homeomorphic processes.

Chapter 4 gives a summary of the basic ideas used to describe non-equilibrium thermodynamics (details appear in Vol 1.).

# Chapter 1
# FALACO SOLITONS

## 1.1 Cosmic Strings in a Swimming Pool

During March of 1986, while visiting an old friend in Rio de Janeiro, Brazil, the present author became aware of a significant topological event involving solitons that can be replicated experimentally by almost everyone with access to a swimming pool. Study the photo on the next page which was taken by David Radabaugh, in the late afternoon, Houston, TX 1986.

The extraordinary photo is an image of the 3 pairs of what are now called Falaco Solitons. The Falaco Soliton consists of a pair of globally stabilized rotational indentations in the free water-air surface of the swimming pool, and an (unseen in the photograph) interconnecting thread from the vertex of one dimple to the vertex of the other dimple that forms the rotational pair. The fluid motion is a local (non-rigid body) rotation motion about the interconnecting thread. In the photo the actual indentations of the free surface are of a few millimeters at most. The lighting and contrast optics enables the dimpled surface structures to be seen (although highly distorted) above and to the left of the black spots on the bottom of the pool. The experimental details of creating these objects are described below. From a mathematical point of view, the Falaco Soliton is a connected pair of two dimensional topological defects connected by a one dimensional topological defect or thread.

The Falaco soliton is easily observed in terms of the black spots associated with the surface indentations. The black circular discs on the bottom of the pool are created by Snell refraction of sunlight on the dimpled surfaces of negative Gauss curvature. Also the vestiges of mushroom spirals in the surface structures around each pair can be seen. The surface spiral arms can be visually enhanced by spreading chalk dust on the free surface of the pool.


**Figure 1. Three pairs of FALACO SOLITONS,** a few minutes after creation. The kinetic energy and the angular momentum of each pair of vortex structures created in the free surface of water quickly decay into dimpled, locally unstable, singular surfaces that have an extraordinary lifetime of many minutes in a still pool. These singular surfaces are connected by means of a stabilizing invisible singular thread, or string, which if abruptly severed will cause the endcaps to disappear in a rapid, non-diffusive manner. The black discs are formed on the bottom of the pool by Snell refraction of a rotationally induced dimpled surface. Careful examination of contrast in the photo will indicate the region of the dimpled surface as deformed artifacts to the left of each black spot at a distance about equal to the separation distance of the top right pair and elevated above the horizon by about 25 degrees. The photo was taken in late afternoon. The fact that the projections are circular and not ellipses indicates that the dimpled surface is a minimal surface.

The 1986 photo ( by David Radabaugh, Schlumberger, Houston) demonstrates the existence of Falaco Solitons, a few minutes after creation, by a mechanism to be described below. The kinetic energy and the angular momentum initially given to a pair of vortex structures created in the free surface of water quickly decay into dimpled, locally unstable, singular surfaces of negative Gauss curvature that can have an extraordinary lifetime of more than 15 minutes in a still pool. These "solitons" are extraordinary for they are examples of solitons that can be created easily in a macroscopic environment. Very few examples of such long-lived topological structures can be so easily created in dynamical systems. Note that if the initial vortex structures were Rankine vortices, the air water interface surfaces would have central regions of positive Gauss curvature. Prior to the dynamic stimulation, the Gauss curvature of the water surface is zero. During the formation phase, the Gauss curvature of the surface may have a positive component, but the vortex structures are definitely not perfect Rankine vortices of positive Gauss curvature (indicating solid body rotation). After the first few seconds of creation, the stabilized surface definitely has negative Gauss curvature, with a Mean curvature of zero. The resulting dimple is a minimal surface.

A more recent photo (2004) indicates vividly the compact nature of the surface defects, and the circular appearance of the black spots refracted to the bottom of the pool. Note that the refractions projected on the bottom of the pool are circular, not ellipsoidal


Experimentally, it is apparent that the "bulk" vortex structures are induced initially by tangential discontinuities of the Kelvin-Helmholtz type in the bulk fluid [195] [197], implying the existence of topological torsion at the discontinuity. The surface phenomena has the characteristic shape of the Rayleigh-Taylor mushroom.

The surface defects of the Falaco Soliton are observed dramatically due the formation of circular black discs on the bottom of the swimming pool. The very dark black discs are emphasized in contrast by a bright ring or halo of focused light surrounding the black disc. All of these visual effects can be explained by means of the unique optics of Snell refraction from a surface of negative Gauss curvature. (This explanation was reached on the day, and about 30 minutes after, the present author became aware of the Falaco effect, while standing under a brilliant Brazilian sun and in the white marble swimming pool of his friend in Rio de Janeiro. An anecdotal history of the discovery is described below.) The dimpled surface created appears to be (almost) a minimal surface with negative Gauss curvature and mean curvature $X_M = 0$. This conclusion is justified by the fact that the Snell projection to the floor of the pool is almost conformal, preserving the circular appearance of the black disc, independent from the angle of solar incidence. (Notice that the black spots on the bottom of the pool in the photo are circular and not distorted ellipses, even though the solar elevation is less than 30 degrees.) The conformal projection property is a property of normal projection from minimal surfaces [229].

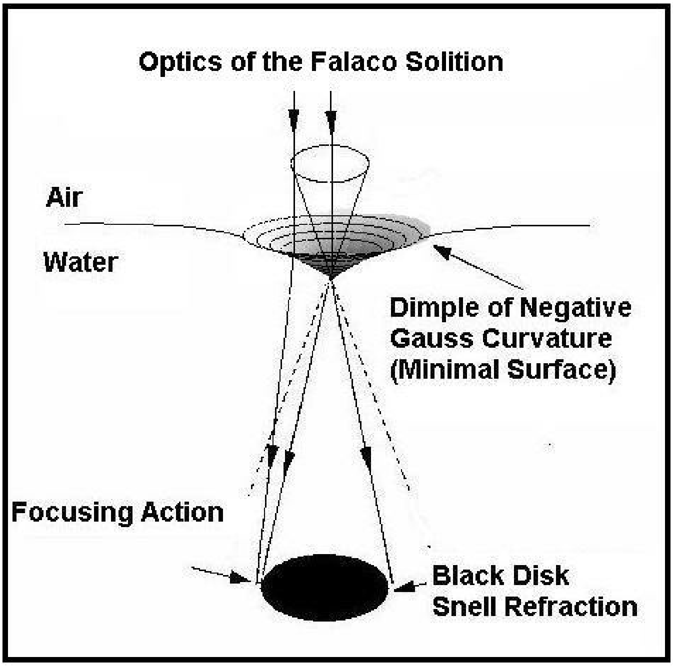



**Figure 2. Optics of the Falaco Soliton**

The effect is easily observed, for in strong sunlight the convex hyperbolic indentation will cause an intensely black circular disk (or absence of light) to be imaged on the bottom of the pool. In addition a bright ring of focused light will surround the black disk, emphasizing the contrast. During the initial few seconds of decay to the metastable soliton state, the large black disk is decorated with spiral arm caustics, remindful of spiral arm galaxies. The spiral arm caustics contract around the large black disk during the stabilization process, and ultimately disappear when the soliton state is achieved. It should be noted that if chalk dust is sprinkled on the surface of the pool during the formative stages of the Falaco soliton, then the topological signature of the familiar Mushroom Spiral pattern is exposed. The black disk optics are completely described by Snell refraction from a surface of revolution that has negative Gauss curvature.

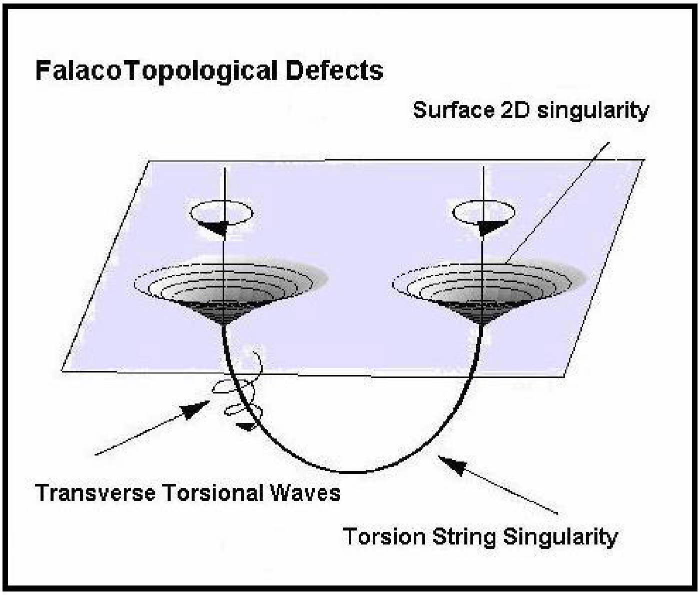



**Figure 3. Falaco Topological Defects.**

Dye injection near an axis of rotation during the formative stages of the Falaco Soliton indicates that there is a unseen thread, or 1-dimensional string singularity, in the form of a circular arc that connects the two 2-dimensional surface singularities or dimples. Transverse Torsional waves of dye streaks can be observed to propagate, back and forth, from one dimple vertex to the other dimple vertex, guided by the "string" singularity. The effect is remindful of the whistler propagation of electrons along the guiding center of the earth's magnetic field lines.

A feature of the Falaco Soliton [178] that is not immediately obvious is that it consists of a pair of two dimensional topological defects, in a surface of fluid discontinuity, which are *connected* by means of a topological singular thread. It is conjectured that the tension in the singular connecting thread provides the force that maintains the global stability of the pair of locally unstable, dimpled surface structures. The equilibrium mode for the free surface requires that the surface be flat, of zero Gauss curvature, without dimples. If dye drops are injected into the water near the rotational axis, and during formative stages of the Falaco Soliton, the dye particles will execute a *torsional* wave motion that oscillates up and down, back and forth, until the dye maps out the thread singularity (a circular arc!) that connects the two vertices of the Falaco Soliton. The singular thread acts as a guiding center for the torsion waves. If the thread is severed, the endcap singularities disappear almost immediately, and not diffusively.

However, as a soliton, the topological system retains its coherence for remarkably long time - more than 15 minutes in a still pool. The long lifetime of the Falaco Soliton is due to this *global stabilization* of the connecting string singularity, even though the surface of negative Gauss curvature is locally unstable. The long life of the soliton state in the presence of a viscous media indicates that the flow vector field describing the dynamics is probably harmonic. This result is in agreement with the assumption that the fluid can be represented by a Navier-Stokes equation with a dissipation that is represented by the product of a viscosity and the vector Laplacian of the velocity field. If the velocity field is harmonic, the vector Laplacian vanishes, and the dissipation goes to zero no matter what the magnitude is of the viscosity term. Hence a palatable argument is offered for the long lifetime. More over it is known that minimal surfaces are generated by harmonic vector fields, hence the minimal surface endcaps give further credence to the idea of a harmonic velocity field.

The bottom line is that it is possible to produce, hydrodynamically, in a viscous fluid with a surface of discontinuity, a long lived coherent structure that consists of a set of macroscopic topological defects. The Falaco Solitons are representative of non-equilibrium long lived structures, or "stationary states", far from equilibrium. These observation were first reported at the 1987 Dynamics Days conference in Austin, Texas [178], [179], and subsequently in many other places, mostly in the hydrodynamic literature [186], [187], [197], [194], [199], as well as at several APS meetings.

These, long-lived topologically coherent objects, dubbed the Falaco Solitons (for reasons explained below), have several features equivalent to those reported for models of the sub-microscopic hadron. String theorists take note, for the structure consists of a pair of topological 2-dimensional locally unstable rotational defects in a surface of discontinuity, globally connected and globally stabilized in the fluid by a 1 dimensional topological defect or string with tension. (The surface defects are of negative Gauss curvature, and are therefor locally unstable.) As mentioned above the experimental equilibrium state is a surface of zero Gauss curvature. However, the local instability is overcome globally by a string whose tension globally stabilizes the locally unstable endcaps. These observational conjectures have now been explained theoretically in terms of a bifurcation process, which is explained in detail below. Note that like hadrons, the endcaps represent the quarks which suffer a confinement problem, for when the confining string is severed, the hadrons (endcaps) disappear.

The reader must remember that the Falaco Soliton is a topological object that can and will appear at all scales, from the microscopic, to the macroscopic, from the sub-submicroscopic world of strings connecting branes, to the cosmological level of spiral arm galaxies connected by cosmic strings. At the microscopic level, the method offers a view of forming spin pairs that is different from Cooper pairs and could offer insight into Superconductivity. At the level of Cosmology, the concept of Falaco Solitons could lead to explanations of the formation of flat spiral arm galaxies. At the submicroscopic level, the Falaco Solitons mimic quarks on a string. At the macroscopic level, the topological features of the Falaco Solitons can be found in solutions to the Navier-Stokes equations in a rotating frame of reference. Under deformation of the discontinuity surface to a flattened ball, the visual correspondence to hurricane structures between the earth surface and the tropopause is remarkable. In short, the concept of Falaco Solitons appears to be a universal phenomena.

#### *1.1.1 The Experiment*

The Falaco Soliton phenomena is easily reproduced by placing a large circular disc with a sharp edge, such as dinner plate, vertically into the swimming pool until the plate is half submerged and it oblate axis resides in the water-air free surface. Then move the plate slowly in the direction of its oblate axis. At the end of the stroke, smoothly extract the plate (with reasonable speed) from the water, imparting kinetic energy and distributed angular momentum to the fluid. Initially, the dynamical motion of the edges of the plate will create a pair of vortex structures in the free surface of the water (a density discontinuity which can also be mimicked by salt concentrations). If these vortex structures were Rankine vortices of opposite rotation, they would cause the initially flat surface of discontinuity to form a pair of parabolic concave indentations of positive Gauss curvature, indicative of the "rigid body" rotation of a pair of contra-rotating vortex cores of uniform vorticity. Observations have not confirmed that the initial vortex structures are Rankine. In any case, in a few seconds the vortex surface depressions will decay into a pair of convex dimples of negative Gauss curvature. Associated with the evolution is a visible set of spiral arm caustics, As the convex dimples form, the surface effects can be observed in bright sunlight via their Snell projections as large black spots on the bottom of the pool. In a few tries you will become an expert experimentalist, for the drifting spots are easily created and, surprisingly, will persist for many minutes in a still pool. The dimpled depressions are typically of the order of a few millimeters in depth, but the zone of circulation typically extends over a disc of some 10 to 30 centimeters or more, depending on the plate diameter. This configuration, or coherent topological defect structure, has been defined as the Falaco Soliton. For purposes of illustration , the vertical depression has been greatly exaggerated in Figures 2 and 3.

If a thin broom handle or a rod is placed vertically in the pool, and the Falaco soliton pair is directed in its translation motion to intercept the rod symmetrically, as the soliton pair comes within range of the scattering center, or rod, (the range is approximately the separation distance of the two rotation centers) the large black spots at first shimmer and then disappear. Then a short time later, after the soliton has passed beyond the interaction range of the scattering center, the large black spots coherently reappear, mimicking the numerical simulations of soliton coherent scattering. For hydrodynamics, this observation firmly cements the idea that these objects are truly coherent "Soliton" structures. This is the only (known to this author) macroscopic visual experiment that replicates the coherence features of soliton scattering as seen in numerical studies.

If the string connecting the two endcaps is sharply "severed", the confined, two dimensional endcap singularities do not diffuse away, but instead disappear almost explosively. It is this observation that leads to the statement that the Falaco soliton is the macroscopic topological equivalent of the illusive hadron in elementary particle theory. The two 2-dimensional surface defects (the quarks) are bound together by a string of confinement, and cannot be isolated. The dynamics of such a coherent structure is extraordinary, for it is a system that is globally stabilized by the presence of the connecting 1-dimensional string. For almost twenty years, challenges to the theoretical string community to devise an explanation for the Falaco Solitons remained unanswered. Now it appears that the method of continuous topological evolution and non-equilibrium thermodynamics is able to demonstrate the generation of Falaco Solitons as solutions to the Navier - Stokes equations.

### 1.2 Falaco Solitons and Dynamical Systems

The objective is to find the proper format for the functional form of the 1-form of Action $A$ that represents a physical system which supports the experimentally long-lived, non-equilibrium topological defects defined as Falaco Solitons. Once the Action 1-form, $A$, is formulated, then the theory of Pfaff Topological dimension can be used to extract topological information and the possibility of irreversible topological evolution.

From the Jacobian matrix of the functional coefficients of the 1-form of Action, similarity invariants can be determined that lead to universal thermodynamic phase properties of the physical system. In this specific case a solution (a 1-form of Action) is desired that replicates the topological properties of the Falaco Soliton. The observable properties are:

1. Two rotational 2D structures that appear to be catenoidal minimal surface defects at the density discontinuity interface.
2. A 1D structure, or thread, that appears to be connected to the vertices of the two dimples of the two surface defects.
3. A relatively long lived state created by irreversible processes, and which appears to be globally stabilized by the connecting string tension.

In classical theories, a model of a physical system is often made in terms of a dynamical system of ordinary differential equations. The question then arises: if given a dynamical system, what is the corresponding 1-form of Action? Unfortunately, a procedure for extracting the 1-form of Action from a dynamical system is not unique. In that which follows, two different methods will be discussed, both of which yield interesting facts about the properties of Falaco Solitons.

### *1.2.1 Bifurcations*

From the observational evidence it is apparent that the Falaco Solitons are *not* locally stable. The endcaps are of negative Gauss curvature which immediately implies that they are close to, if not exactly equal to, minimal surfaces. Any solution that models the Falaco Solitons must replicate such structures. All real minimal surfaces are locally unstable, as the Jacobian matrix has both positive and negative real roots. However, like a soap film between two rings (the Catenoid minimal surface) boundary conditions can stabilize the system, globally. This global stabilization may exist only for a limited range. As a homework problem [112] p.235, Landau demonstrates that the minimal soap film between two rings of equal diameter is globally stabilized as long as the separation of the rings is less that ~1.33 times the ring diameter. This effect has been demonstrated in a solution to the Navier-Stokes equation [187]. If the boundary conditions (the rings) are not too far apart the catenoid soap film of zero mean curvature and negative Gauss curvature has a long persistent lifetime. The catenoid minimal surface has one component. If the boundary conditions are too far apart, the catenoid bifurcates into two components, where both the Gauss curvature and the Mean curvature are zero.

For soap films it is should be remembered that the actual thin membrane consists of two superposed surfaces, containing an interior media. Motion pictures (with a fast frame rate) of the bifurcation dynamics of a soap film between two rings indicate that as the rings are slowly separated the bifurcation condition is reached and subsequently the catenoid double surface contracts to a conical "focus" shape, and then separates into two cones. The two cones continue to contract ultimately becoming the flat films across each ring. This process has been demonstrated in exact solutions to the Navier-Stokes equations which undergo a Saddle-Node Hopf bifurcation [187]. The "catenoid" in the fluid is defined by the implicit surface where the function of fluid helicity density vanishes, $\Phi = \mathbf{v} \cdot curl\ \mathbf{v} \Rightarrow 0$.

### *1.2.2 Projective Fluctuations and dynamical systems*

The first method of deducing a 1-form of Action from a dynamical system will be based upon the dualism of projective geometry. The second method (detailed below) will be to utilize the Hopf map, which is a non-linear time dependent projection. To repeat some of the experimental observation, recall that apparently the Falaco Solitons are globally stabilized by a "thread of surface tension", or string, that connects the two dimples. If this "thread" is severed, the endcaps do not diffuse away but decay immediately. It will be demonstrated that the dynamical system that mimics the production of the Falaco Solitons is a solution to the Navier-Stokes equations for a swirling fluid. In effect, the Falaco Soliton is a bifurcation to a long lived topological defect in a dissipative medium.

In this section the Falaco Solitons will be considered to be produced by a dynamical system extended to include topological fluctuations. The vector field $\mathbf{V}$ is an evolutionary field, but need not be a "kinematically perfect" velocity field. In short, consider the Falaco Solitons to be represented by an "extended" dynamical system, topologically equivalent to an exterior differential system of 1-forms,

$$\omega^x = dx - \mathbf{V}^x(x,y,z,t)dt \Rightarrow 0, \quad (1.1)$$
$$\omega^y = dy - \mathbf{V}^y(x,y,z,t)dt \Rightarrow 0, \quad (1.2)$$
$$\omega^z = dz - \mathbf{V}^z(x,y,z,t)dt \Rightarrow 0. \quad (1.3)$$

When all three 1-forms vanish, imposing the existence of a topological limit structure on the base manifold of 4 dimensions, $\{x,y,z,t\}$, the result is equivalent to a 1D solution manifold defined as a (perfect) kinematic system. The solution manifold to the dynamical system is in effect a parametrization of the parameter $t$ to the space curve $C_{parametric}$ in 4D space, where for kinematic perfection, $[\mathbf{V}^k,1]$ is a tangent vector to the curve $C_{parametric}$. Off the kinematic solution submanifold, the non-zero values for the 1-forms, $\omega^k$, can be interpreted as topological fluctuations from "kinematic perfection".

If "kinematic perfection" is not exact, then the three 1-forms $\omega^k$ are not precisely zero, and have a finite non-zero triple exterior product that defines a $N-1=3$ form in the 4 D space. From the theory of exterior differential forms it is the intersection of the zero sets of these three hypersurfaces $\omega^k$ that creates an implicit curve $C_{implicit}$ in 4D space.

$$\begin{aligned} C_{implicit} &= \omega^x\hat{}\omega^y\hat{}\omega^z && (1.4)\\ &= dx\hat{}dy\hat{}dz - \mathbf{V}^x dy\hat{}dz\hat{}dt + \mathbf{V}^y dx\hat{}dz\hat{}dt - \mathbf{V}^z dx\hat{}dy\hat{}dt \\ &= -i([\mathbf{V},1])\Omega_4. && (1.5)\end{aligned}$$

The discussion brings to mind the dualism between points (rays) and hypersurfaces (hyperplanes) in projective geometry.

If a ray (a "point" in a the projective 3 space of 4 dimensions) is specified by the 4 components of a the 4D vector $[\mathbf{V},1]$ multiplied by any non-zero factor, $\kappa$, (such that $[\mathbf{V},1] \approx \kappa[\mathbf{V},1]$), then the equation of a dual (or adjoint) projective hyperplane is given by the expression $[\mathbf{A},-\phi]$ such that

$$\langle\gamma[\mathbf{A},-\phi]| \circ |\kappa[\mathbf{V},1]\rangle = 0. \quad (1.6)$$

The principle of projective duality [149] implies that (independent from the factors $\gamma$ and $\kappa$)

$$\phi = \mathbf{A}\circ\mathbf{V}. \quad (1.7)$$

A particularly easy choice is to assume that (to within a factor)

$$\begin{aligned}
\mathbf{A}_k &= \mathbf{V}^k, \text{ and } \quad \phi = \mathbf{V} \circ \mathbf{V}, & (1.8)\\
A &= V_k dx^k - V_k V^k dt. & (1.9)\\
V_k(x,y,z,t) &\equiv V^k(x,y,z,t), & (1.10)
\end{aligned}$$

where $V^k(x,y,z,t)$ are the 3 functions of the dynamical system.

Consider a dynamical system that has an adjoint projection that can be encoded (to within a factor, $1/\lambda$) on the variety of independent variables $\{x,y,z,t\}$ in terms of a 1-form of Action:

$$A = \{A_k(x,y,z,t)dx^k - \phi(x,y,z,t)dt\}/\lambda(x,y,z,t). \tag{1.11}$$

Note that it is presumed that this 1-form of Action has terms which have the same physical dimension. In the general case, the coefficient functions are canonical conjugates of the differentials of the independent variables. In thermodynamics, the coefficients are presumed to be homogeneous of degree 0, and are called intensive variables, while the differential functions, are presumed to be homogeneous of degree 1, and are call extensive variables. In special cases (of interest herein) the coefficients, $A_k$, are "dimensionless" functions of the independent variables. For these special cases, it is possible to first renormalize the coefficient functions by a choice of $\lambda$, then construct the Jacobian matrix of the (renormalized covariant) coefficient functions:

$$[\mathbb{J}_{jk}(A)] = \left[\partial(A_j/\lambda)/\partial x^k\right]. \tag{1.12}$$

This Jacobian matrix can be interpreted as a projective correlation mapping of "points" (contravariant vectors) into "hyperplanes" (covariant vectors). The correlation mapping is the dual of a collineation mapping, $\left[\mathbb{J}(\mathbf{V}^k)\right]$, which takes points into points. Linear (local) stability occurs at points where the (possibly complex) eigenvalues of the Jacobian matrix, $[\mathbb{J}_{jk}(A)]$, are such that the real parts are not positive. The eigenvalues, $\xi_k$, are determined by solutions to the Cayley-Hamilton characteristic polynomial of the Jacobian matrix, $[\mathbb{J}(A)]$:

$$\Theta(x,y,z,t;\ \xi) = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi + T_K \Rightarrow 0. \tag{1.13}$$

As the elements of the Jacobian matrix are presumed to be real, then the similarity coefficients are real. In such cases, the Cayley-Hamilton polynomial equation defines a family of implicit functions in the space of real variables, $X_M(x, y, z, t)$, $Y_G(x, y, z, t)$, $Z_A(x, y, z, t)$, $T_K(x, y, z, t)$. The functions $X_M$, $Y_G$, $Z_A$, $T_K$ are defined as the similarity invariants of the Jacobian matrix. If the eigenvalues, $\xi_k$, are distinct, then the similarity invariants are given by the expressions:

$$
\begin{aligned}
& \text{Linear} \\
X_M &= \xi_1 + \xi_2 + \xi_3 + \xi_4 = Trace\,[\mathbb{J}_{jk}]\,, && (1.14) \\
& \text{Quadratic} \\
Y_G &= \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1 + \xi_4\xi_1 + \xi_4\xi_2 + \xi_4\xi_3, && (1.15) \\
& \text{Cubic} \\
Z_A &= \xi_1\xi_2\xi_3 + \xi_4\xi_1\xi_2 + \xi_4\xi_2\xi_3 + \xi_4\xi_3\xi_1, && (1.16) \\
& \text{Quartic} \\
T_K &= \xi_1\xi_2\xi_3\xi_4 = \det\,[\mathbb{J}_{jk}]\,. && (1.17)
\end{aligned}
$$

Note that if the vector field $V_k(x, y, z, t)$ is independent from time (autonomous) then the Quartic similarity invariant is always zero, when $\lambda = 1$.

When the coefficients of the 1-form of action are homogeneous in the sense that they have the same physical dimension to within a constant factor, and when the independent variables (in this case $x, y, z, s = ct$) are physical lengths, then eigenvalues are of "reciprocal length" and are defined as "Curvatures". The Similarity Invariants then become linear, quadratic, cubic and quartic compositions of curvatures. For example, the quadratic curvature is then equivalent to the well known Gauss curvature of an implicit hypersurface. If the renormalization (or "expansion") factor, $\lambda$, is the Gauss map:

$$\lambda = (A_x^2 + A_y^2 + A_z^2 + \phi^2)^{1/2}. \qquad (1.18)$$

then Jacobian matrix of the "normalized" vector field becomes the Shape matrix of differential geometry when $\lambda$ is the the Gauss map. The renormalized components of the 1-form of Action are homogeneous of degree 0. It is remarkable that the determinant of the Jacobian matrix constructed from the renormalized vector field is always zero. For a 4 dimensional system, this result implies that one eigenvector of the Jacobian has a zero eigen value. This result does not imply that the Pfaff topological dimension of the Action 1-form is less than 4. The similarity invariant vanishes $T_K = 0$, but the topological parity does not, $K = dA\hat{}dA \neq 0$ . However, the Jacobian matrix of the normalized Action generated the shaped matrix of differential geometry, and the eigenvalues have dimensions of curvature, or inverse extent (like $1/r$).

$$\begin{aligned}
& & \text{Cayley Hamilton Similarity Invariants} \quad (1.19)\\
& & \text{of a Normalized Action 1-form}\\
& & \text{Linear Term}\\
X_M &\Rightarrow & \text{Mean Curvature} = Trace\,[\mathbb{J}_{jk}]\,, \quad (1.20)\\
& & \text{Quadratic Term}\\
Y_G &\Rightarrow & \text{Gauss Curvature,} \quad (1.21)\\
& & \text{Cubic Term}\\
Z_A &\Rightarrow & \text{Adjoint Curvature} \quad (1.22)\\
& & \text{Quartic Term}\\
T_K &\Rightarrow & 0 = \det\,[\mathbb{J}_{jk}]\,. \quad (1.23)
\end{aligned}$$

Note that the 4th similarity invariant is zero for the Gauss map scaling. The Gauss map scaling coefficient is chosen to be the quadratic isotropic Holder norm of index 1. The choice of the Gauss map may be viewed as a projection of the N dimensional variety to an implicit hypersurface of dimension N-1. The similarity invariants can be related to curvatures under the projection, but the choice of the Gauss map is special, for it is a topological constraint forcing the similarity invariant of highest degree, $T_K$, to vanish*. Physical significance can be associated with $T_K \neq 0$, hence herein the objects of interest are the similarity invariants, not the curvatures.

Bifurcation and singularity theory involve the zero sets of the similarity invariants, and the algebraic intersections of the implicit hypersurfaces so generated by these zero sets. Recall that the theory of linear (local) stability requires that the eigenvalues of the Jacobian matrix have real parts which are not greater than zero. For a 4th order polynomial, either all 4 eigenvalues are real; or, two eigenvalues are real, and two eigenvalues are complex conjugate pairs; or there are two distinct complex conjugate pairs. Local stability therefor requires:

*A non-zero term $T_K$ will be related to the Higgs potential, and the Landau-Ginsberg $\Psi^4$ theory.

**Local Stability**

$$\begin{aligned} X_M &\leq 0, &(1.24)\\ Y_G &\geq 0, &(1.25)\\ Z_A &\leq 0, &(1.26)\\ T_K &\geq 0. &(1.27)\end{aligned}$$

It should be remembered that the Cayley-Hamilton characteristic polynomial in terms of similarity coordinates is a representation of a family of implicit hypersurfaces in a 4D space. The function is

In that which follows, the similarity coefficients of the thermodynamic 1-form, $A$, will be studied for physical systems of various Pfaff topological dimensions. In the next chapter, 1-forms representing turbulent non-equilibrium systems will be studied as candidates for cosmological models. At present, the focus will be directed to interpretations of 1-forms that can be put into correspondence with dynamical systems, and their relationship to Falaco Solitons.

### *1.2.3 Clues from the Hopf Map and Hopf vectors.*

**The Hopf Map and its Adjoint field of Pfaff dimension 4.**

The second method of extracting a 1-form of Action from a given dynamical system (vector field or a system of Pfaffian 1-forms) can be studied using the Hopf map (with modifications) as an example. Again, the Hopf map may be viewed as a projection from a domain of 4 variables to a domain of three variables. If the three variables are interpreted as a velocity field, then the method can define a dynamical system. The Hopf map is a rather remarkable projective map from 4 to 3 (real or complex) dimensions that has interesting and useful topological properties related to links and braids and other forms of entanglement. In this sense, the range of the Hopf map can be viewed as special example of constructing a dynamical system when the three range functions are interpreted as a "velocity" vector field, $[u, v, w]$, or the components of a Pfaffian system $\{\omega^x, \omega^y, \omega^z\}$, As will be demonstrated, the adjoint 1-form to the Hopf map satisfies the criteria of Local Stability, and yet is not an integrable system. The 1-form deduced from the Hopf map is of Pfaff Topological dimension 4, and admits irreversible dissipation for processes in the direction of the Topological Torsion vector (See Vol 1.)

Consider the map from R4[X,Y,Z,S] to R3[u,v,w] given by the formulas

$$\mathbf{H}1 = [u1, v1, w1] = [2(XZ + YS), 2(XS - YZ), (X^2 + Y^2) - (Z^2 + S^2)]. \tag{1.28}$$

The components $[u1, v1, w1]$ can be considered as the velocity components of a dynamical system. These formulas define the format of a Hopf map. The 3 component Hopf vector $\mathbf{H}1$ is real and has the property that

$$\mathbf{H}1 \cdot \mathbf{H}1 = (u1)^2 + (v1)^2 + (w1)^2 = (X^2 + Y^2 + Z^2 + S^2)^2. \tag{1.29}$$

Hence a real (and imaginary) 4 dimensional sphere maps to a real 3 dimensional sphere. If the functions $[u1, v1, w1]$ are defined as $[x/ct, y/ct, z/ct]$, then the 4D sphere $(X^2 + Y^2 + Z^2 + S^2)^2 = 1$, implies that the Hopf map formulas are equivalent to the 4D light cone. Other selections for the ordered pairs of $(X, Y, Z, S)$ (along with permutations of the 3 vector components) give distinctly different Hopf vectors. For example,

$$\mathbf{H}2 = [2(YX - SZ), X^2 + Z^2 - Y^2 - S^2, -2(ZY + SX)], \tag{1.30}$$

is another Hopf vector, a map from R4 to R3, but with the property that $\mathbf{H}2$ is orthogonal to $\mathbf{H}1$ :

$$\mathbf{H}2 \cdot \mathbf{H}1 = 0. \tag{1.31}$$

Similarly, a third linearly independent orthogonal Hopf vector $\mathbf{H}3$ can be found

$$\mathbf{H}3 = [X^2 + Y^2 - Z^2 - S^2, -2(YX + SZ), 2(-ZX + SY)] \tag{1.32}$$

such that

$$\mathbf{H}2 \cdot \mathbf{H}1 = \mathbf{H}3 \cdot \mathbf{H}2 = \mathbf{H}2 \cdot \mathbf{H}3 = 0, \tag{1.33}$$

$$\mathbf{H}1 \cdot \mathbf{H}1 = \mathbf{H}2 \cdot \mathbf{H}2 = \mathbf{H}3 \cdot \mathbf{H}3 = (X^2 + Y^2 + Z^2 + S^2)^2. \tag{1.34}$$

The three linearly independent Hopf vectors can be used as a basis of R3 excluding those points where the quartic form vanishes. The mapping functions $(u, v, w)$ of the Hopf vector can be differentiated with respect to $(X, Y, Z, S)$ to produce a set of three exact 1- form whose coefficients form 3 independent 4 component vectors on R4. A 4th linearly independent vector can be created algebraically by constructing the adjoint matrix to the matrix of 3 independent 4 component vectors. The components of the 4 component adjoint vector can be used to define the Hopf 1-form, $A_{Hopf}$. For **H**1, the 4 independent 1-forms are given by the expressions (where $\lambda(X, Y, Z, S)$ is an arbitrary scaling function):

$$d(u1) = 2Zd(X) + 2Sd(Y) + 2Xd(Z) + 2Yd(S) \tag{1.35}$$
$$d(v1) = 2Sd(X) - 2Zd(Y) - 2Yd(Z) + 2Xd(S) \tag{1.36}$$
$$d(w1) = 2Xd(X) + 2Yd(Y) - 2Zd(Z) - Sd(S) \tag{1.37}$$

$$A_{Hopf} = \{-Yd(X) + Xd(Y) - Sd(Z) + Zd(S)\}/\lambda \tag{1.38}$$

This direction field can be used to construct a non-integrable 1-form, $A$, of Pfaff dimension 4. It is some interest to examine the properties of the adjoint 1-form, $A_{Hopf}$, defined hereafter as the Hopf 1-form. For $\lambda = 1$, it follows that the Hopf 1-form is of Pfaff dimension 4. The exterior derivatives of the basis frame produce the usual Cartan connection which is not affine-torsion free in its subspaces. By this mechanism the differential structure of R4 as induced by the Hopf map is determined.

The Hopf map can be viewed as a dynamical system with velocity components $[X, Y, Z, S] \Rightarrow [U, V, W]$, or the map can be viewed as a projective non-linear map from a position vector in 4D to a position vector in 3D, $[X, Y, Z, S] \Rightarrow [x, y, z]$. In the position vector interpretation, the differentials of the components of the position vector, $[dX, dY, dZ]$ can be used to define an affine mapping and topological fluctuations of kinematic perfection.

Properties of the Hopf map include:

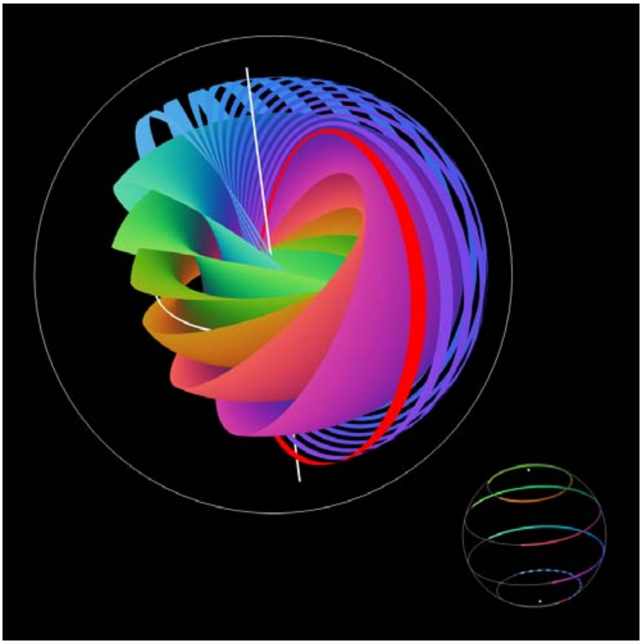
1. The Hopf map is a map from 4 to 3 (real or complex) dimensions that has interesting and useful topological properties related to links and braids and other forms of entanglement.

2. The Hopf map is a projection which can be used to determine a global basis frame for the variety in terms of 3 exact 1-forms and 1 adjoint 1-form which is of Pfaff dimension 4. The Frame field so defined has non-zero affine torsion.

3. The Hopf adjoint field can be used to represent, within a factor, the 1-form of Action (potentials) for a certain class of electromagnetic fields that exhibit propagating non-zero topological torsion and non-zero topological spin.

4. The Hopf map yields two pairs of orthogonal 3 vectors, one which is left-handed and the other which is right handed. The 4 form of topological parity, $dA\hat{}dA$ constructed from the respective adjoint fields is either negative or positive.

5. The complex sum of two Hopf vectors generates a Cartan spinor.



**Visualization of the Hopf Map created by Ken Shoemake**

One of the interesting features of the unscaled Hopf 1-form is that it is of Pfaff topological dimension 4, and satisfies the criteria of Local Stability. Of particular interest is that the unscaled Hopf Map, and the unit normalized (Gauss) Hopf map are locally stable

**The Chirality clue**

The formula for the Hopf 1-form, $A$, (given in eq. (**??**)) can be generalized to include two constant coefficients of chirality, $\Omega$ and $\Gamma$, to read

$$A_{Hopf} = \{\Omega \cdot (-Yd(X) + Xd(Y)) + \Gamma \cdot (-Sd(Z) + Zd(S))\}/\lambda. \quad (1.39)$$

The chiral coefficients, at first, will be presumed to be constants, but can have any finite value, positive or negative. Each chiral pair, $(-Yd(X) + Xd(Y)$ and $(-Sd(Z)+Zd(S))$, can have the same or opposite chirality sense depending on the signs of $\Omega$ and $\Gamma$. The combination has the 4D appearance of linked rinks. A 3D projected image of the linkages is given in the preceeding figure.

For $\lambda = 1$, it follows that the Hopf 1-form is of Pfaff dimension 4 with the topological torsion 4 vector,

$$\mathbf{T}_4 = 2 \cdot \Omega \cdot \Gamma \cdot [X, Y, Z, S]. \quad (1.40)$$

and with a "dissipation" coefficient,

$$\begin{aligned} &\text{Hopf Topological Parity} && (1.41)\\ K &= dA_{Hopf}\hat{}dA_{Hopf} && (1.42)\\ &= 8 \cdot \Omega \cdot \Gamma \cdot \{dX\hat{}dY\hat{}dZ\hat{}dS\} \lessgtr 0. && (1.43) \end{aligned}$$

As the sign of the Topological Parity 4-form determines whether or not the 4D volume element is expanding or contracting, it follows that the relative chirality sense of the two links is physically measurable.

The Jacobian matrix of the coefficients of the Hopf 1-form (for $\lambda = 1$) becomes

$$JAC_{Hopf} := \begin{bmatrix} 0 & -\Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Gamma \\ 0 & 0 & \Gamma & 0 \end{bmatrix}, \quad (1.44)$$

with eigenvalues $e1 = \sqrt{-1}\Omega$, $e2 = -\sqrt{-1}\Omega$, $e3 = \sqrt{-1}\Gamma$, $e4 = -\sqrt{-1}\Gamma$, and with similarity invariants,

$$\begin{aligned} \text{Hopf} &: \text{Similarity Invariants} \\ X_M &= 0 \leq 0, && (1.45) \\ Y_G &= \Omega^2 + \Gamma^2 \geq 0, && (1.46) \\ Z_A &= 0 \leq 0, && (1.47) \\ T_K &= \Omega^2 \Gamma^2 \geq 0. && (1.48) \end{aligned}$$

Hence, from the theory of dynamical systems, the canonical Hopf 1-form is locally stable. Note that both the Hopf Topological Parity, $K$, and the Topological Torsion vector, $\mathbf{T}_4$, depend upon the sense of the two chirality coefficients, $\Omega$ and $\Gamma$, but the similarity invariants do not. The topological properties are chiral sensitive to antisymmetries, where the similarity coefficients are not. If the chirality coefficients of the two Hopf rings are of the same sense, then the dissipation coefficient is related to a contraction of the 4D volume. If the chirality coefficients are of opposite sign, then the dissipation coefficient corresponds to a 4D expansion.

As mentioned earlier, it is also of interest to consider factors $\lambda$ that are of the format of the Holder norm, where n and p are integers, and $(a, b, k, m)$ are arbitrary constants:

$$\lambda = (aX^p + bY^p + kZ^p + mS^p)^{n/p}. \tag{1.49}$$

The exponents n and p determine the homogeneity of the resulting 1-form, which is given below with an ambiguous format $(R, L)$ depending on the relative chirality of the two Hopf links:

$$A_{RL} = A/\lambda = \{\Omega(Yd(X) - Xd(Y)) + \Gamma(-Sd(Z) + Zd(S))\}/\lambda. \tag{1.50}$$

For example, for $n = 1, p = 2$, and arbitrary constants, $(a, b, k, m)$, the determinant of the Jacobian matrix vanishes, implying that at least one eigenvalue is zero. It follows that $T_K = 0$, but remarkably the Pfaff Topological dimension of $A_{Hopf}/\lambda$ remains equal to 4. If the coefficients $(a, b, k, m)$ are all equal to unity, then $\lambda$ is in effect the Gauss map, and the Jacobian of the normalized Hopf 1-form is the implicit form of the Shape Matrix. The result demonstrates that $T_K = 0$ does not imply that $K = div\ \mathbf{T}_4 = 0$.

For arbitrary n, p, and $(a, b, k, m)$, the 3-form of topological (Hopf) torsion becomes:

$$Topological\ Torsion = (A)\hat{}d(A) = i(\mathbf{T}_4)d(X)\hat{}d(Y)\hat{}d(Z)\hat{}d(S), \tag{1.51}$$

where the topological torsion 4 vector is equal to:.

$$\mathbf{T}_4 = 2 \cdot \Omega \cdot \Gamma \cdot [X, Y, Z, S]/\lambda^2. \tag{1.52}$$

The Torsion vector, $\mathbf{T}_4$, for the Hopf map is proportional to the position vector from the four dimensional origin and represents an expansion or a contraction process. The factor $\Lambda$ depends upon the integers n and p as well as the constants ( a,b,k,m).

The Topological Parity 4-form, whose coefficient is the 4 divergence of the Torsion vector, $\mathbf{T}_4$, becomes

$$\begin{aligned} & Topological\ Parity && (1.53)\\ = & \ d(A)\hat{}(d(A) && (1.54)\\ = & \ 4(2-n) \cdot \Omega \cdot \Gamma \cdot d(X)\hat{}d(Y)\hat{}d(Z)\hat{}d(S)/\lambda^2. && (1.55)\end{aligned}$$

It is most remarkable that for n=2, any p and any (a,b,k,m), and any relative chirality, the topological parity of the rescaled 1-form vanishes; the resulting scaled Hopf 1-form is of Pfaff Topological dimension 3, not 4. In such cases the ratios of the integrals of the topological torsion 3 form over various closed manifolds are rational, and the closed integrals of the 3-form are topological deformation invariants (coherent structures). The topological torsion and the topological parity are sensitive to the relative senses of $\Omega$ and $\Gamma$.

Next, for simplicity of expression consider the isotropic case, $a = 1,\ b = 1,\ k = 1,\ m = 1,\ p = 2$ and compute the similarity invariants. These special choices, especially for the quadratic exponent $p = 2$ cause both $X_M$ and $Z_A$ to vanish. (Other non-isotropic choices and other values for p can force the linear and cubic similarity invariants to be positive.) Note that the similarity invariants do not depend upon the relative senses of $\Omega$ and $\Gamma$.

$$\lambda = (\Omega^2(X^2+Y^2) + \Gamma^2(Z^2+S^2))^{n/2} \tag{1.56}$$

$$X_M = 0 \leq 0, \tag{1.57}$$

$$\begin{array}{ll} n=0 & Y_G = \Omega^2 + \Gamma^2 \geq 0, \\ n=1 & Y_G = 1/\lambda^2 \geq 0 \\ n=2 & Y_G = 0 \end{array} \tag{1.58}$$

$$Z_A = 0 \leq 0, \tag{1.59}$$

$$\begin{array}{ll} n=0 & T_K = \Omega^2 \cdot \Gamma^2/\lambda^4 \geq 0 \\ n=1 & T_K = 0 \geq 0 \\ n=2 & T_K = -\Omega^2 \cdot \Gamma^2/\lambda^4 \leq 0. \end{array} \tag{1.60}$$

Note that for n=2, the Pfaff topological dimension of the renormalized Hopf Action 1-form is 3, not for, as the $d(A\hat{}dA) = 0$. However, the quartic similarity invariant $T_K$ is not zero and is negative, but the quadratic similarity invariant $Y_G = 0$. For n = 1, $d(A\hat{}dA)$ is not zero (and is positive). The quartic similarity invariant $T_K = 0$, but the quadratic similarity invariant is positive and not zero. The odd similarity invariants vanish in both cases. For n = 0 and n = 1, the conditions of local stability are satisfied. For n = 2, the system is not locally stable.

The clues obtained from this study of Hopf maps focuses attention on the concept of chirality components which cannot be ignored in the general solution for a 1-form of Action, $A$. Moreover, the concept of chirality sense is not contained in the similarity invariants (the coefficients enter as squares) but is contained in the expressions for the Topological Torsion and Topological Parity components of the non-equilibrium system. The moral of the story is that not all physical properties are included in symmetries.

**Spinors as linear combinations of Hopf Maps**

It is also important at this point to realize that complex combinations of Hopf vectors can be combined to form a 3D isotropic (null) complex position vector, $[z1, z2, z3]$. The resulting real and an imaginary components have the same magnitude and are orthogonal. In short, the Cartan Spinor [39] can be represented as the complex sum of two Hopf vectors. The spinors come in two triples of the form:

$$|\boldsymbol{\sigma}_{12}\rangle = |\mathbf{H}1\rangle + i\ |\mathbf{H}2\rangle \quad \textit{with} \quad \langle\boldsymbol{\sigma}_{12}| \circ |\boldsymbol{\sigma}_{12}\rangle = 0 \tag{1.61}$$
$$|\boldsymbol{\sigma}_{23}\rangle = |\mathbf{H}2\rangle + i\ |\mathbf{H}3\rangle \quad \textit{with} \quad \langle\boldsymbol{\sigma}_{23}| \circ |\boldsymbol{\sigma}_{23}\rangle = 0 \tag{1.62}$$
$$|\boldsymbol{\sigma}_{31}\rangle = |\mathbf{H}3\rangle + i\ |\mathbf{H}1\rangle \quad \textit{with} \quad \langle\boldsymbol{\sigma}_{31}| \circ |\boldsymbol{\sigma}_{31}\rangle = 0. \tag{1.63}$$

These complex combinations of Hopf vectors can be used to generate solutions for which the Topological Torsion vanishes, and yet the Topological Spin† is finite and quantized [164] [250]. The spinor combinations also can be used to generate conjugate minimal surfaces, as will be demonstrated in the next chapter.

It should be remembered that not all dynamic features are captured by the similarity invariants of a dynamic system . The antisymmetric features of the dynamics is better encoded in terms of Cartan's magic formula.

†Topological Spin is another 3-form distinct from the concept of Topological Torsion.

Cartan's formula expresses the evolution of a 1-form of Action, $A$, in terms of the Lie differential with respect to a vector field, $V$, acting on the 1-form that encodes the properties of the physical system. For example, consider the 1-form of Action (the canonical form of a Hopf system) given by the equation (with a change of notation)

$$A = \Omega \cdot (-ydx + xdy) + \Gamma \cdot (-tdz + zdt). \tag{1.64}$$

The Jacobian matrix of this Action 1-form has eigenvalues which are solutions of the characteristic equation,

$$\Theta(x, y, z, t;\ \xi)_{Hopf} = (\xi^2 + \Omega^2)(\xi^2 + \Gamma^2) \Rightarrow 0. \tag{1.65}$$

The eigenvalues are two conjugate pairs of pure imaginary numbers, $\{i\Omega, -i\Omega\}$ and $\{i\Gamma, -i\Gamma\}$ and are interpreted as "oscillation" frequencies. The similarity invariants are $X_M = 0$, $Y_G = \Omega^2 + \Gamma^2 > 0$, $Z_A = 0$, $T_K = \Omega^2\Gamma^2 > 0$. The Hopf eigenvalues have no real parts that are positive, and so the Jacobian matrix is locally stable. The criteria for a double Hopf oscillation frequency requires that the algebraically odd similarity invariants vanish and the algebraically even similarity invariants are positive definite. The stability critical point of the Hopf bifurcation occurs when all similarity invariants vanish. In such a case the oscillation frequencies are zero. This Hopf critical point is NOT necessarily the same as the thermodynamic critical point, as exhibited by a van der Waals gas. The oscillation frequencies have led the Hopf solution to be described as a "breather". The Hopf system is a locally stable system in four dimensions. Each of the pure imaginary frequencies can be associated with a "minimal" hypersurface.

Suppose that $\Gamma \Rightarrow 0$ such that $Z_A = 0$, $T_K = 0$ . Then the resulting characteristic equation represents a "strange minimal surface" in the sense that $X_M = 0$, but with a Gauss curvature which is positive definite, $Y_G = \Omega^2 > 0$. The curvatures of the implicit surface are imaginary. In differential geometry, where the eigenfunctions can be put into correspondence with curvatures, the Hopf condition, $X_M = 0$, for a single Hopf frequency would be interpreted as "strange" minimal surface (attractor ?). The surface would be strange for the condition $Y_{G(hopf)} = \Omega^2 > 0$ implies that the Gauss curvature for such a minimal surface is positive. A real minimal surface has curvatures which are real and opposite in sign, such that the Gauss curvature is negative.

As a real minimal surface has eigenvalues with one positive and one negative real number, the criteria for local stability is not satisfied for real minimal surfaces. Yet experience indicates that soap films can occur as "stationary states". The implication is that soap films can be globally stabilized, even though they are locally unstable.

As developed in the next section, the Falaco critical point and the Hopf critical point are the same: all similarity invariants vanish. For the autonomous examples it is possible to find an implicit surface, $Y_{G(hopf)} = Y_{G(falaco)} = 0$, in terms of the variables $\{x, y, z;\ A, B, C...\}$ where $A, B, C...$ are the parameters of the dynamical system.

Recall that the classic (real) minimal surface has real curvatures with a sum equal to zero, but with a Gauss curvature which is negative $(X_M = 0, Y_G < 0)$. Such a system is not locally stable, for there exist eigenvalues of the Jacobian matrix with positive real parts. Yet persistent soap films exist under such conditions and are apparently stable macroscopically (globally). This experimental evidence can be interpreted as an example of global stability overcoming local instability.

### *1.2.4 The bifurcation to Falaco Solitons*

Similar to and guided by experience with the Hopf bifurcation, the bifurcation that leads to Falaco Solitons must agree with the experimental observation that the endcaps have negative Gauss curvature, and are in rotation. The stability of the Falaco Soliton is global, experimentally, for if the singular thread connecting the vertices is cut, the system decays non-diffusively. Hence the bifurcation to the Falaco Soliton can not imply local stability. This experimental result is related to the theoretical confinement problem in the theory of quarks. To analyze the problem consider the case where the $T_K$ term in the Cayley-Hamilton polynomial vanishes (implying that one eigenvalue of the 4D Jacobian matrix is zero). Experience with the Hopf bifurcation suggests that Falaco Soliton may be related to another form of the characteristic polynomial, where $X_M = 0,\ Z_A = 0,\ Y_G < 0$. This bifurcation is not equivalent to the Hopf bifurcation, but has the same critical point, in the sense that all similarity invariants vanish at the critical point. Similar to the Hopf bifurcation this new bifurcation scheme can be of Pfaff topological dimension 4, which implies that the abstract thermodynamic system generated by the 1-form (which is the projective dual to the dynamical system) is an open, non-equilibrium thermodynamic system. The odd similarity invariants of the 4D Jacobian matrix must vanish. However there are substantial differences between the bifurcation that lead to Hopf solitons (breathers) and Falaco solitons. Experimentally, the Falaco soliton appears to have a projective cusp at the critical point (the vertex of the dimple) and that differs from the Hopf bifurcation which would be expected to have a projective parabola at the critical point.

When $T_K = 0$, the resulting cubic factor of the characteristic polynomial can have 1 real eigenvalue, $b$, and 1 pair of complex conjugate eigenvalues, $(\sigma + i\Omega), (\sigma - i\Omega)$. To be stable globally it is presumed that

$$\text{Global Stability } X_M = b + 2\sigma \leq 0, \tag{1.66}$$

$$Z_A = b(\sigma^2 + \Omega^2) \leq 0, \tag{1.67}$$

$$\text{with } Y_G = \sigma^2 + \Omega^2 + 2b\sigma \text{ undetermined.} \tag{1.68}$$

If all real coefficients are negative then $Y_G > 0$, and the system is locally stable. Such is the situation for the Hopf bifurcation. However, the Falaco Soliton experimentally requires that $Y_G < 0$.

By choosing $b \leq 0$, in order to satisfy $Z_A \leq 0$, leads to the constraint that $\sigma = -b/2 > 0$, such that the real part of the complex solution is positive, and represents an expansion, not a contraction. Substitution into the formula for $Y_G$ leads to the condition for generation of a Falaco Soliton:

$$Y_{G(falaco)} = \Omega^2 - 3b^2/4 < 0. \tag{1.69}$$

It is apparent that local stability is lost for the complex eigenvalues of the Jacobian matrix can have positive real parts, $\sigma > 0$. Furthermore it follows that $Y_G < 0$ (leading to negative Gauss curvature) if the square of the rotation speed, $\Omega$, is smaller than 3/4 of the square of the real (negative) eigen value, $b$. This result implies that the "forces" of tension overcomes the inertial forces of rotation. In such a situation, a real minimal surface is produced (as visually required by the Falaco soliton). The result is extraordinary for it demonstrates a global stabilization is possible for a system with one contracting direction, and two expanding directions coupled with rotation. The contracting coefficient $b$ (similar to a spring constant) is related to the surface tension in the "string" that connects the two global endcaps of negative Gaussian curvature. The critical point occurs when $\Omega^2 = 3b^2/4$.

It is conjectured that if the coefficient $b$ is in some sense a measure of of a reciprocal length (such that $b$ ˜ $1/R$, a curvature), then there are three interesting formulas comparing angular velocity (orbital period) and length (orbital radius).

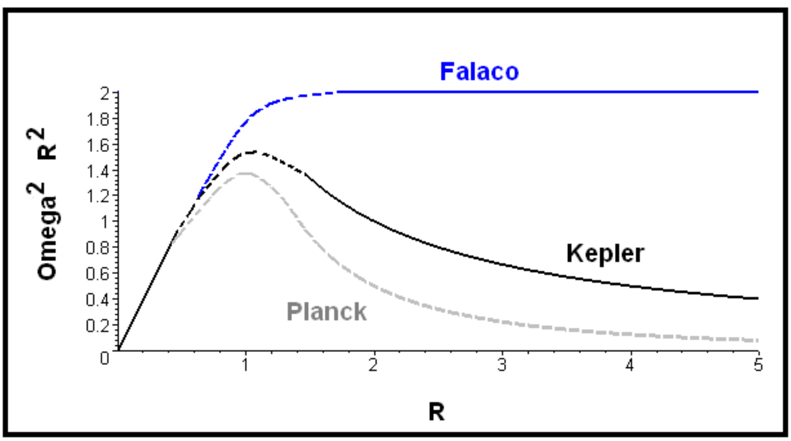
$$\text{Falaco : } \quad \Omega^2 R^2 = \text{constant} \tag{1.70}$$

$$\text{Kepler : } \quad \Omega^2 R^3 = \text{constant} \tag{1.71}$$

$$\text{Planck : } \quad \Omega^2 R^4 = \text{constant.} \tag{1.72}$$

The bifurcations to Hopf Solitons suggest oscillations of expansions and contractions of imaginary minimal surfaces (or Soliton concentration breathers) and have been exhibited in the certain chemical reactions such as the Besalouv - Zhabotinski system. On the other hand, the bifurcations to Falaco Solitons suggest the creation of spiral concentrations, or density waves, on real rotating minimal surfaces. The molal density distributions (or order parameters) are complex. The visual bifurcation structures of the Falaco Solitons in the swimming pool would appear to offer an explanation as to the origin of (~flat) spiral arm galaxies at a cosmological level, and would suggest that the spiral arm galaxies come in pairs connected by a topological string. Moreover, the kinetic energy of the stars far from the galactic center would not vary as the radius of the "orbit" became very large. This result is counter to the Keplerian result that the kinetic energy of the stars should decrease as 1/R.

If is assumed that the density distribution of star mass is more or less constant over the central region of the spiral arm flat disc-like structures, then over this region, the Newtonian gravitation force would lead to a "rigid body" result, $\Omega^2 R^2 = R$. If it is assumed that the density distribution then decreases dramatically in the outer regions of the spiral arms, then it has been assumed that Keplerian formula holds. The following Figure demonstrates the various options:




This result will be mentioned again in the next chapter, where a Cosmological theory is developed in terms of non-equilibrium defect structures embedded in a turbulent dissipative very dilute non-equilibrium extension of a van der Waals gas.

## 1.3 Falaco Solitons in exact solutions to the Navier-Stokes equations.

The idea that multiple parameter Dynamical Systems can produce tertiary bifurcations was studied by Langford [117]. His developments were organized about certain non-linear equations in polar coordinates, with multiple parameters $(r, \theta, z, t; A, B, , ..)$.

$$dr/dt = rg(r, z, A, G, C) \tag{1.73}$$
$$d\theta/dt = 1 \tag{1.74}$$
$$dz/dt = f(r, z, A, B, D) \tag{1.75}$$

It is remarkable that these tertiary bifurcations can be demonstrated to be solutions of the Navier-Stokes equations in a rotating frame of reference [191]. Langford was interested in how these "normal" forms of dynamical systems could cause bifurcations to Hopf breather-solitons. Herein, it is also of interest to determine how and where these dynamical systems can cause bifurcations to Falaco rotational solitons.

It is of some pedagogical utility to transform the Langford equations to $\{x, y, z\}$ coordinates with parameters, $A, B, ....$ In polar coordinates, a map between the variables $\{x, y, z\} \Rightarrow \{r, \theta, z\}$ leads to the following expressions:

$$r = \sqrt{x^2 + y^2}, \qquad dr = (xdx + ydy)/\sqrt{x^2 + y^2}, \tag{1.76}$$
$$\theta = \tan(y/x), \qquad d\theta = \pm(ydx - xdy)/(x^2 + y^2) \tag{1.77}$$
$$z = z, \qquad dz = dz \tag{1.78}$$

Substitution of the differentiable map into the polar equations yields the system of 1-forms

$$\omega^1 = dr - rg(r, z, A, B...)dt \tag{1.79}$$
$$= (xdx + ydy)/\sqrt{x^2 + y^2} - rg(r, z, A, B...)dt \tag{1.80}$$
$$\omega^2 = d\theta - \Omega dt = \pm(ydx - xdy)/(x^2 + y^2) - \omega dt \tag{1.81}$$
$$\omega^3 = dz - f(r, z, A, B)dt. \tag{1.82}$$

The 3-form $C$ composed from the three 1-forms becomes to within an arbitrary factor,

$$\begin{aligned} \omega^1\hat{}\omega^2\hat{}\omega^3 &= i(\rho\mathbf{V}_4)\Omega_4 = i(\rho\mathbf{V}_4)dx\hat{}dy\hat{}dz\hat{}dt, \qquad (1.83) \\ &= -\{\mathbf{V}^x dy\hat{}dz\hat{}dt - \mathbf{V}^y dx\hat{}dz\hat{}dt + \mathbf{V}^z dx\hat{}dy\hat{}dt - dx\hat{}dy\hat{}dt\}, \\ \text{where } \mathbf{V}_4 &= [\mathbf{V}_3, 1]. \qquad (1.84) \end{aligned}$$

with

$$\begin{aligned} \mathbf{V}^x &= \{\mp\Omega y + (xg(r,z,A,B...\} \qquad (1.85) \\ \mathbf{V}^y &= \{\pm\Omega x + (yg(r,z,A,B...)\} \qquad (1.86) \\ \mathbf{V}^z &= f(r,z,\lambda,\alpha) \qquad (1.87) \end{aligned}$$

The rotation speed (angular velocity) is represented by $\Omega$. The Langford examples are specializations of the functions $f(r,z,A,B...)$ and $g(r,z,A,B...)$. The following examples yield solutions to the similarity invariants for three of the Langford examples that he described as the Saddle Node Hopf bifurcation, the Hysteresis Hopf bifurcation, and the Transcritical Hopf bifurcation.

The similarity invariants are computed for the projective dual 1-form,

$$\text{Projective dual 1-form } A = V_k dx^k - V_k V^k dt. \qquad (1.88)$$

It can be shown that the Pfaff topological dimension of the projective dual 1-form of each of the examples below is 4. This fact implies that the abstract thermodynamic system is an open system far from thermodynamic equilibrium [247]. Thermodynamic systems of Pfaff dimension 4 are inherently dissipative, and admit processes which are thermodynamically irreversible. Such irreversible processes are easily computable, and are proportional to the 3-form of Topological Torsion, $A\hat{}dA$. If

$$\begin{aligned} \text{for } i(\mathbf{T}_4)\Omega_4 &= A\hat{}dA, \qquad (1.89) \\ \text{such that } \langle\mathbf{T}_4| \circ |\mathbf{V}_4\rangle &= 0, \qquad (1.90) \end{aligned}$$

then the dynamical system is *not* representative of an irreversible process. For a physical system represented by the projective dual 1-form, and a process defined by the direction field of the dynamical system, the process is irreversible only if it has a component in the direction of the Topological Torsion vector. Direct computation indicates that for the projective dual 1-form composed of the components of an autonomous dynamical system, then a necessary condition for reversibility in an abstract thermodynamic sense is that the helicity density must vanish. Hence a sufficient condition for irreversibility is

**Theorem 1** *Autonomous processes* $\mathbf{V}$ *(such that* $\partial\mathbf{V}/\partial t = 0$*) are irreversible if* $\mathbf{V} \circ curl\ \mathbf{V} \neq 0$

### *1.3.1 Saddle-Node: Hopf and Falaco bifurcations*

The dynamical system

$$
\begin{aligned}
f &= A + Bz^2 + D(x^2 + y^2) && (1.91)\\
g &= (G + Cz) && (1.92)\\
dx/dt &= \mathbf{V}^x = x(G + Cz) \mp \Omega y && (1.93)\\
dy/dt &= \mathbf{V}^y = y(G + Cz) \pm \Omega x && (1.94)\\
dz/dt &= \mathbf{V}^z = A + Bz^2 + D(x^2 + y^2) && (1.95)
\end{aligned}
$$

Similarity Invariants for $A = V_k dx^k - V^k V_k dt$

$$
\begin{aligned}
X_M &= 2(G + Cz + Bz) && (1.96)\\
Y_G &= +\Omega^2 - 2CD(x^2 + y^2) + G^2 + 2G(C + 2B)z \\
&\quad +C^2(1 + 4B)z^2 && (1.97)\\
Z_A &= +\{2Bz\}\Omega^2 + 2(G + Cz)(GBz + CBz^2 \\
&\quad -DC(x^2 + y^2) && (1.98)\\
T_K &= 0 && (1.99)
\end{aligned}
$$

The similarity invariants are also chiral invariants with respect to the sign of the rotation parameter, $\Omega$. The criteria for Hopf oscillations requires that $X_M = 0$, and $Z_A = 0$. When these constraints are inserted into the formula for $Y_G$ they yield $Y_{G(hopf)}$. The criteria for oscillations is that $Y_{G(hopf)} > 0$.

$$
\text{Hopf Constraint} \quad Y_{G(hopf)} = +3\Omega^2 - B^2 z^2 > 0, \qquad (1.100)
$$

$$
\text{Oscillation frequencies} \quad : \quad \omega = \pm\sqrt{Y_{G(hopf)}} \qquad (1.101)
$$

Note that $Y_{G(hopf)}$ is a quadratic form in terms of the rotation parameter. When $Y_{G(hopf)} < 0$, it is defined as $Y_{G(falaco)}$. It is therefor easy to identify the tension parameter, $b$, for the Falaco Soliton by evaluating the Falaco formula

$$Y_{G(falaco)} = \Omega^2 - 3b^2/4 < 0. \tag{1.102}$$

$$\text{Falaco tension } b^2 = 4B^2z^2/9. \tag{1.103}$$

The coefficient $b$ can be interpreted as the Hooke's law force (tension) associated with a linear spring extended in the $z$ direction, with a spring constant equal to $2/3B$. Indeed, computer solutions to the Saddle node Hopf system indicate the trajectories can be confined internally to a sphere, and that Falaco surfaces of negative Gauss curvature are formed at the North and South poles by the solution trajectories.

$$\text{Helicity} = \mathbf{V} \circ curl\ \mathbf{V} \tag{1.104}$$

$$Helicity = -(C(x^2 + y^2) + 2A + 2Bz^2)\Omega$$

If the process described by the dynamical system is to be reversible in a thermodynamic sense, then the Helicity must vanish. This constraint fixes the value of the rotation frequency $\Omega$ in the autonomous system for reversible bifurcations.

The Hopf-Falaco critical point in similarity coordinates can be mapped to an implicit surface in xyz coordinates, eliminating the rotation parameter, $\Omega$.

$$Y_{G(hopf-critical)} = Y_{G(falaco-critical)} = -(3DC(x^2 + y^2) + 4B^2z^2) \Rightarrow 0. \tag{1.105}$$

Depending on the values assigned to the parameters, and especially the signs of $C$ and $D$, the critical surface is either open or closed. When the critical surface function is positive, the Hopf-Falaco bifurcation leads to Hopf Solitons (breathers), and if the critical surface function is negative, the bifurcation leads to Falaco Solitons.

### *1.3.2 Hysteresis-Hopf and Falaco bifurcations*

$$f = A + Bz + Ez^3 + D(x^2 + y^2) \tag{1.106}$$

$$g = (-G + Cz) \tag{1.107}$$

$$dx/dt = \mathbf{V}^x = x(-G + Cz) \mp \Omega y \tag{1.108}$$

$$dy/dt = \mathbf{V}^y = y(-G + Cz) \pm \Omega x \tag{1.109}$$

$$dz/dt = \mathbf{V}^z = A + Bz + Ez^3 + D(x^2 + y^2) \tag{1.110}$$

Similarity Invariants for $A = V_k dx^k - V^k V_k dt$

$$
\begin{aligned}
X_M &= 2(Cz + G) + (B + 3Ez^2) && (1.111)\\
Yg &= \Omega^2 + \{G^2 - 2GB - 2DC(x^2 + y^2)\} + \{2G(B - C)\}z \\
&\quad + \{C^2 - 6GE\}z^2 + \{6CE\}z^3 && (1.112)\\
Z_A &= \{B + 3Ez^2\}\Omega^2 + \{2GCD(x^2 + y^2) + G^2 B\} + \\
&\quad \{-2C^2 D(x^2 + y^2) - 2GCB\}z + \{3G^2 E + C^2 B\}z^2 \\
&\quad + \{-6GCE\}z^3 + \{3C^2 E\}z^4 && (1.113)\\
T_K &= 0 && (1.114)
\end{aligned}
$$

The criteria for Hopf oscillations requires that $X_M = 0$, and $Z_A = 0$. When these constraints are inserted into the formula for $Y_G$ they yield $Y_{G(hopf)}$. The criteria for oscillations is that $Y_{G(hopf)} > 0$.

$$
\begin{aligned}
\text{Hopf Condition } Y_{G(hopf)} &= 3\Omega^2 - 3/2(Ez^2)(3/2(Ez^2) + B) - 1/4B^2 > 0. && (1.115)\\
\text{Oscillation frequency } \omega &= \pm\sqrt{Y_{G(hopf)}}. && (1.116)
\end{aligned}
$$

Note that (like the Saddle Node Hopf case) $Y_{G(hopf)}$ is a quadratic form in terms of the rotation parameter. It is therefor easy to identify the tension parameter for the Falaco Soliton by evaluating the Falaco formula

$$
\begin{aligned}
Y_{G(falaco)} &= \Omega^2 - 3b^2/4 < 0, && (1.117)\\
\text{Falaco tension } b^2 &= (9E^2 z^4 + 6BEz^2 + B^2)/9. && (1.118)
\end{aligned}
$$

In this case the tension is not that of a linear spring, but instead can be interpreted as a non-linear spring constant for extensions in the $z$ direction. Indeed, computer solutions to the Hysteresis - Hopf - Falaco system indicate the trajectories can be confined internally to a sphere-like surface, and that Falaco minimal surfaces are visually formed at the North and South poles [117].

$$\text{Helicity} = \mathbf{V} \circ curl\ \mathbf{V}$$

$$Helicity = -\{C(x^2 + y^2) + 2A + 2z(B + Ez^2)\}\Omega.$$

If the process described by the dynamical system is to be reversible in a thermodynamic sense, then the Helicity must vanish. This constraint fixes the value of the rotation frequency $\Omega$ in the autonomous system for reversible bifurcations.

The Hopf-Falaco critical point in similarity coordinates can be mapped to an implicit surface in xyz coordinates, eliminating the rotation parameter, $\Omega$.

$$Y_{G(hopf-critical)} = Y_{G(falaco-critical)} = -(3DC(x^2+y^2)+(3EZ^2+B)^2) \Rightarrow 0. \tag{1.119}$$

Depending on the values assigned to the parameters, and especially the signs of $C$ and $D$, the critical surface is either open or closed. When the critical surface function is positive, the Hopf-Falaco bifurcation leads to Hopf Solitons (breathers), and if the critical surface function is negative, the bifurcation leads to Falaco Solitons. Note that if $E = 0$, $DC < 0$, then there is a circular limit cycle in the x,y plane. Direct integration of the differential equations demonstrates the decay to this attractor.

### *1.3.3 Transcritical Hopf and Falaco Bifurcations*

The dynamical system

$$\begin{aligned}
f &= Az + Bz^2 + D(x^2 + y^2)) && (1.120)\\
g &= A - G + Cz && (1.121)\\
dx/dt &= \mathbf{V}^x = x(A - G + Cz) \mp \Omega y && (1.122)\\
dy/dt &= \mathbf{V}^y = y(A - G + Cz) \pm \Omega x && (1.123)\\
dz/dt &= \mathbf{V}^z = Az + Bz^2 + D(x^2 + y^2) && (1.124)
\end{aligned}$$

Similarity Invariants for $A = V_k dx^k - V^k V_k dt$

$$\begin{aligned}
X_M &= 3A - 2G + 2(C + B)z && (1.125)\\
Yg &= +\Omega^2 - 2CD(x^2 + y^2) + (4CB + C^2)z^2 + 2z(2B(A - G) \\
&\quad +C(2C - G) + (G^2 + 3A^2 - 4GA) && (1.126)\\
Z_A &= +\{A + 2Bz\}\Omega^2 + A^3 + 2A^2Bz - 2GA^2 - 4AGBz + 2CzA^2 \\
&\quad +4ACz^2B - 2ACy^2D + G^2A + 2G^2Bz - 2GCzA - 4GCz^2B \\
&\quad +2GCy^2D + C^2z^2A + 2C^2z^3B - 2C^2zy^2D \\
&\quad +2Dx^2C(G - A - Cz) && (1.127)\\
T_K &= 0 && (1.128)
\end{aligned}$$

The similarity invariants are chiral invariants relative to the rotation parameter $\Omega$. The criteria for Hopf oscillations requires that $X_M = 0$, and

$Z_A = 0$. When these constraints are inserted into the formula for $Y_G$ they yield $Y_{G(hopf)}$. The criteria for (breather) oscillations is that $Y_{G(hopf)} > 0$.

$$\text{Hopf Condition } Y_{G(hopf)} = 3\Omega^2 ABz - B^2z^2 - 1/4A^2 > 0 \quad (1.129)$$

$$\text{Oscillation frequency } \omega = \pm\sqrt{-Y_{G(hopf)}} \quad (1.130)$$

Note that (again) $Y_{G(hopf)}$ is a quadratic form in terms of the rotation parameter. It is therefor easy to identify the tension parameter for the Falaco Soliton by evaluating the Falaco formula

$$Y_{G(falaco)} = \Omega^2 - 3b^2/4. \quad (1.131)$$

$$\text{Falaco tension } b^2 = (4B^2z^2 + A^2)/9ABz). \quad (1.132)$$

In this case the tension is again to be associated with a non-linear spring with extensions in the z direction.

$$\text{Helicity} = \mathbf{V} \circ curl\ \mathbf{V}$$

$$H_{bifurcation} = -\{C(x^2 + y^2) + 2z(A + Bz)\}\Omega.$$

If the process described by the dynamical system is to be reversible in a thermodynamic sense, then the Helicity must vanish. This constraint fixes the value of the rotation frequency $\Omega$ in the autonomous system for reversible bifurcations.

The Hopf-Falaco critical point in similarity coordinates can be mapped to an implicit surface in xyz coordinates, eliminating the rotation parameter, $\Omega$.

$$Y_{G(hopf-critical)} = Y_{G(falaco-critical)} = -(3DC(x^2 + y^2) + (2Bz + A)^2) \Rightarrow 0. \quad (1.133)$$

Depending on the values assigned to the parameters, and especially the signs of $C$ and $D$, the critical surface is either open or closed. When the critical surface function is positive, the Hopf-Falaco bifurcation leads to Hopf Solitons (breathers), and if the critical surface function is negative, the bifurcation leads to Falaco Solitons. Note that if $B = 0$, $DC < 0$, then there is a circular limit cycle in the x,y plane. Direct integration of the differential equations demonstrates the decay to this attractor.

### *1.3.4 Minimal Surface Hopf and Falaco Bifurcations*

The utility of Maple becomes evident when generalizations of the Langford systems can be studied.

The dynamical system

$$\begin{aligned}
f &= A + Bz + Fz^2 + Ez^3 + D(x^2 + y^2)) && (1.134)\\
g &= G + Cz && (1.135)\\
dx/dt &= \mathbf{V}^x = x(G + Cz) \mp \Omega y && (1.136)\\
dy/dt &= \mathbf{V}^y = y(G + Cz) \pm \Omega x && (1.137)\\
dz/dt &= \mathbf{V}^z = A + Bz + Fz^2 + Ez^3 + D(x^2 + y^2)) && (1.138)
\end{aligned}$$

can be studied with about as much ease as all of the preceding examples. An especially interesting case is given by the system

$$\begin{aligned}
f &= A + P\sinh(\alpha z) + D(x^2 + y^2)) && (1.139)\\
g &= G + Cz && (1.140)\\
dx/dt &= \mathbf{V}^x = x(G + Cz) \mp \Omega y && (1.141)\\
dy/dt &= \mathbf{V}^y = y(G + Cz) \pm \Omega x && (1.142)\\
dz/dt &= \mathbf{V}^z = a + P\sinh(\alpha z) + D(x^2 + y^2)) && (1.143)
\end{aligned}$$

Similarity Invariants for $A = V_k dx^k - V^k V_k dt$

$$\begin{aligned}
X_M &= 2(G + Cz) + \alpha P\cosh(\alpha z) && (1.144)\\
Yg &= +\Omega^2 - 2CD(x^2 + y^2) + (G + Cz)^2\\
&\quad +2(G + Cz)P\alpha\cosh(\alpha z) && (1.145)\\
Z_A &= (+\Omega^2 + (G + Cz)^2)P\alpha\cosh(\alpha z)\\
&\quad -2CD(G + Cz)(x^2 + y^2) && (1.146)\\
T_K &= 0 && (1.147)
\end{aligned}$$

The similarity invariants are chiral invariants relative to the rotation parameter $\Omega$. The criteria for Hopf oscillations requires that $X_M = 0$, and $Z_A = 0$. When these constraints are inserted into the formula for $Y_G$ they yield $Y_{G(hopf)}$. The criteria for (breather) oscillations is that $Y_{G(hopf)} > 0$.

$$\text{Hopf Condition } Y_{G(hopf)} = 3\Omega^2 - 1/4\alpha^2 P^2 (\cosh(\alpha z))^2 > 0 \quad (1.148)$$

$$\text{Oscillation frequency} : \omega = \pm\sqrt{-Y_{G(hopf)}} \quad (1.149)$$

Note that (again) $Y_{G(hopf)}$ is a quadratic form in terms of the rotation parameter. It is therefor easy to identify the tension parameter for the Falaco Soliton by evaluating the Falaco formula

$$Y_{G(falaco)} = \Omega^2 - 3b^2/4. \quad (1.150)$$

$$\text{Falaco tension } b^2 = (\alpha^2 P^2 (\cosh(\alpha z))^2)/3). \quad (1.151)$$

In this case the tension is again to be associated with a non-linear spring with extensions in the z direction.

$$\text{Helicity} = \mathbf{V} \circ curl\ \mathbf{V}$$

$$H_{bifurcation} = -\{C(x^2 + y^2) + 2(A + P\sinh(\alpha z))\}\Omega.$$

If the process described by the dynamical system is to be reversible in a thermodynamic sense, then the Helicity must vanish. This constraint fixes the value of the rotation frequency $\Omega$ in the autonomous system for reversible bifurcations.

The Hopf-Falaco critical point in similarity coordinates can be mapped to an implicit surface in xyz coordinates, eliminating the rotation parameter, $\Omega$.

$$Y_{G(hopf\_critical)} = Y_{G(falaco\_critical)} \quad (1.152)$$

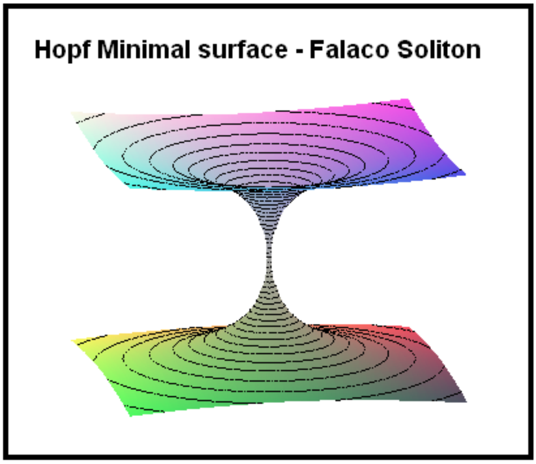
$$= -\{3DC(x^2 + y^2) + \alpha^2 P^2 (\cosh(\alpha z))^2\} \Rightarrow 0 \quad (1.153)$$

When the parameters $DC$ have a product which is negative, then the critical surface is the catenoid – *A Minimal Surface*. That is the Hopf critical surface is an implicit surface of given by the equation,

$$(x^2 + y^2) = \{(\alpha^2 P^2)/(3|DC|)\}(\cosh(\alpha z))^2 \quad (1.154)$$

The throat diameter of the catenoid is proportional to the coefficient

$$Diam = 2\sqrt{(\alpha^2 P^2)/(3|DC|)}. \quad (1.155)$$




The dissipation coefficient due to a non-zero divergence of the Topological Torsion 3-form is

$$K = 8 \cdot \Omega\{(x^2 + y^2) \cdot (C^2 z + GC + PDcosh(\alpha z)\alpha) \tag{1.156}$$
$$+Pcosh(\alpha z)\alpha(a + Psinh(\alpha z))\} \tag{1.157}$$

This minimal surface solution to the Navier-Stokes equations mimics some of the minimal surface features of Falaco Solitons. By adding a singularity to the dynamical system the extent of the minimal surfaces can be made compact.

## 1.4 Falaco Solitons as Landau Ginsburg structures in micro and mesoscopic systems

The Falaco experiments lead to the idea that such topological defects are available at all scales. Consider the possibility in the microphysical domain that is governed by the Landau - Ginsburg theory. With a change of notation ($\xi \Rightarrow \Psi$), the Universal Phase function, eq(1.13 ), can be solved for the similarity invariant $T_K$,

$$T_K = -\{\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi\}. \tag{1.158}$$

The similarity invariant $T_K$ represents the determinant of the Jacobian matrix. All determinants are, in effect, N - forms on the domain of independent variables. All N-forms can be related to the exterior derivative of some N-1 form or current, $J$. Hence

$$dJ = T_K\Omega_4 = (div\mathbf{J}+\partial\rho/\partial t)\Omega_4 = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi)\Omega_4. \quad (1.159)$$

For currents of the form

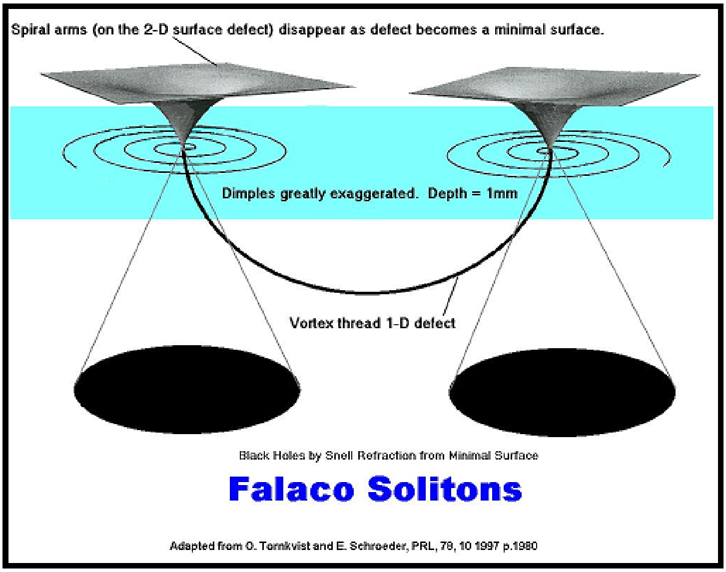
$$\mathbf{J} = grad\ \Psi, \quad (1.160)$$
$$\rho = \Psi, \quad (1.161)$$

the Universal Phase function generates the universal Ginsburg Landau equations

$$\nabla^2\Psi + \partial\Psi/\partial t = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi). \quad (1.162)$$

The work of the previous chapter which applied the concepts of the universal phase function to the study of Falaco Solitons suggests a strong connection between Falaco Solitons and Landau-Ginsburg theory. The Falaco Solitons consist of spiral "vortex defect" structures (analogous to CGL theory) on a two dimensional minimal surface, one at each end of a 1-dimensional "vortex line" or thread (analogous to GPG theory). Remarkably the topological defect surface structure is locally unstable, as the surface is of negative Gauss curvature. Yet the pair of locally unstable 2-D surfaces is *globally* stabilized by the 1-D line defect attached to the "vertex" points of the minimal surfaces.

For some specific physical systems it can be demonstrated that period (circulation) integrals of the 1-form of Action potentials, $A$, lead to the concept of "vortex defect lines". The idea is extendable to "twisted vortex defect lines" in three dimensions. The "twisted vortex defects" become the spiral vortices of a Complex Ginsburg Landau (CGL) theory , while the "untwisted vortex lines" become the defects of Ginzburg-Pitaevskii-Gross (GPG) theory [236].




For super cold Bose Einstein condensates, the rotation defect structures have been described as U-shaped vortex singularities with dimples on the ends of the vortex "lines". the rotational minimal surfaces of negative Gauss curvature which form the two endcaps of the Falaco soliton, like quarks, apparently are confined by the string. If the string (whose "tension" induces global stability of the unstable endcaps) is severed, the endcaps (like unconfined quarks in the elementary particle domain) disappear (in a non-diffusive manner). In the microscopic electromagnetic domain, the Falaco soliton structure offers an alternate, topological, pairing mechanism on a Fermi surface, that could serve as an alternate to the Cooper pairing in superconductors.

In the macroscopic domain, the experiments visually indicate "almost flat" spiral arm structures during the formative stages of the Falaco solitons. In the cosmological domain, it is suggested that these universal topological defects represent the ubiquitous "almost flat" spiral arm galaxies. In the next chapter, the conjecture is made that the cosmological universe is indeed a configuration of topological defects associated with the mass fluctuations of a universal (deformable) van der Waals gas near its critical point.

## 1.5 Summary

As the Falaco phenomena appears to be the result of a topological defect, it follows that as a topological property of hydrodynamic evolution, it could appear in any density discontinuity, at any scale. This rotational pairing mechanism, as a topological phenomenon, is independent from size and shape, and could occur at both the microscopic and the cosmic scales. In fact, as mentioned above, during the formative stages of the Falaco Soliton pair, the decaying vortex structures exhibit spiral arms easily visible as caustics emanating from the boundary of each vortex core. The observation is so striking that it leads to the conjecture: Can the nucleus of M31 be connected to the nucleus of our Milky way galaxy by a tubular cosmic thread? Can material be ejected from one galaxy to another along this comic thread? Can barred spirals be Spiral Arm galaxies at an early stage of formation - the bar being and exhibition of material circulating about the stabilizing thread? At smaller scales, the concept also permits the development of another mechanism for producing spin-pairing of electrons in the discontinuity of the Fermi surface, or in two dimensional charge distributions. Could this spin pairing mechanism, depending on transverse wave, not longitudinal wave, coupling be another mechanism for explaining superconductivity? As the defect is inherently 3-dimensional, it must be associated with a 3-form of Topological Torsion, A^dA, introduced by the author in 1976 [170] [186] [187] [194], but now more commonly called a Chern Simons term, when applied to properties of a linear connection. These ideas were exploited in an attempt to explain high TC superconductivity [189]. To this author the importance of the Falaco Solitons is that they offer the first clean experimental evidence of topological defects taking place in a dynamical system. Moreover, the experiments are fascinating, easily replicated by anyone with access to a swimming pool, and stimulate thinking in almost everyone that observes them, no matter what his field of expertise. They certainly are among the most easily produced solitons.

## 1.6 Some Anecdotal History

Just at the end of WW II, one of my first contacts at MIT was a Brazilian young man named Jose Haraldo Hiberu FALCAO. He was in metallurgy and I was in physics. We became close friends and roommates during the period 1946-1950. He spent much of his time chasing the girls and playing soccer for MIT. Now MIT is not known for its athletic achievements, and when one weekend Haraldo scored two goals - giving MIT one of its few wins (ever) - the sports section of one of the Boston papers, misspelled his name with the headline ~

"FALACO SCORES TWO GOALS - MIT WINS"

Frankly I do not remember the exact headline from more than 55 years ago, but one thing is sure: Haraldo FALCAO was known as FALACO ever since.

Haraldo moved back to Brazil and our ways parted. I became interested in many things, the most pertinent to this story included topological defects and topological evolution in physical systems. In 1986 I thought it would be great fun to go to Rio to see my old college friend, and then go to Machu Pichu to watch Haley's comet go by. My son was an AA pilot, so as parents we got a free Airline Ticket ticket to Brazil. Haraldo had married into a very wealthy family and had constructed a superb house that his wife had designed, hanging onto a cliff-side above Sao Coronado beach south of Rio. Haraldo had a white marble swimming pool next to the house fed by a pristine stream of clear water.

The morning after my wife and I arrived in Rio (Haraldo's chauffeur met us at the airport in a big limo) I got up, after sleeping a bit late, and went to the pool for a morning dip. Haraldo and his family had gone to work, and no one was about. I sat in the pool, wondering about the fortunes of life, and how Haraldo - who I had help tutor to get through MIT - was now so very wealthy, and here I was - just a *pauvre* university professor. I climbed out of the pool, and was met by two servants who had been waiting in the wings. One handed me a towel and a terry cloth robe, and the other poured coffee and set out some breakfast fruit, croissants, etc.

I put a lot of sugar into the strong Brazilian coffee, as Haraldo had taught me to do long ago, and was stirring the coffee when I turned toward the pool (about 5-10 minutes after climbing out of the pool). In the otherwise brilliant sunshine, two black disks (about 15 cm in diameter ) with bright halo rings were slowing translating along the pool floor. The optics caught my attention. Is there something about the southern hemisphere that is different? Does the water go down the drain with a different rotation? What was the cause of these Black Discs?

I went over to the pool, jumped in to investigate what was going on, and Voila!!!, the black discs disappeared. I thought: Here was my first encounter of the third kind and I blew it.

I climbed out of the pool, again, and then noticed that a pair of what I initially thought to be Rankine vortices was formed as my hips left the water, and that these rotational surfaces (which would be surface depressions of positive Gauss curvature if they were Rankine vortex structures) decayed within a few seconds into a pair of rotational surfaces of negative Gauss curvature. Each of the ultimate rotational surfaces were as if someone had depressed slightly a rubber sheet with a pencil point, forming a dimple. As the negative Gauss curvature surfaces stabilized, the optical black disks were formed on the bottom of the pool. The extraordinary thing was that the surface deformations, and the black spots, lasted for some 15 minutes !!!. They were obviously rotational solitons.

The rest is history, and is described on my website and in several published articles in some detail. The first formal presentation was at the 1987 Austin Dynamic Days get together, where my presentation and photos cause quite a stir. The Black Discs were quickly determined to be just an artifact of Snell's law of refraction of the solar rays interacting with the dimpled surfaces of negative Gauss curvature. I conjectured that this Soliton was a topological defect, which caused the mathematicians to take note. It was then that I met Dennis Sullivan, who many years later, along with Bobenko, would influence me again with the concept that minimal surfaces and spinors were related ideas.

What was not at first apparent in the swimming pool experiment was that there is a circular "string" – a 1D topological defect – that connects the two 2D topological defects of negative Gauss curvature. The string extends from one dimple to the other. The string becomes evident if you add a few drops of dye to the water near the rotation axis of one of the "dimples". Moreover, experimentation indicated that the long term soliton stability was due to the global effect of the "string" connecting the two dimpled rotational surfaces. If the arc is sharply severed, the dimples do not "ooze" away, as you would expect from a diffusive process; instead they disappear quite abruptly. It startled me to realize that the Falaco Solitons have the confinement properties (and problems) of two quarks on the end of a string.

I called the objects FALACO SOLITONS, for they came to my attention in Haraldo's pool in Rio. Haraldo will get his place in history. I knew that finally I had found a visual, easily reproduced, experiment that could be used to show people the importance and utility of Topological Defects in the physical sciences, and could be used to promote my ideas of Continuous Topological Evolution.

The observations were highly motivating. The experimental observation of the Falaco Solitons greatly stimulated me to continue research in applied topology, involving topological defects, and the topological evolution of such defects which can be associated with phase changes and thermodynamically irreversible and turbulent phenomena. When colleagues in the physical and engineering sciences would ask “What is a topological defect?” it was possible for me to point to something that they could replicate and understand visually at a macroscopic level.

The topological ideas have led ultimately to

1. A non-statistical method of describing processes that are thermodynamically irreversible.
2. Applications of Topological Spin and Topological Torsion in classical and quantum field theories.
3. Another way of forming Fermion pairs
4. A suggestion that spiral galaxies may be stabilized by a connecting "thread", and an explanation of the fact that stars in the far reaches of galactic spiral arms do not obey the Kepler formula.
5. A number of patentable ideas in fluids, electromagnetism, and chemistry.

The original observation was first described at a Dynamics Days conference (1987) in Austin, TX, [178] and has been reported, as parts of other research, in various hydrodynamic publications, but it is apparent that these concepts have not penetrated into other areas of research. As the phenomena is a topological defect, and can happen at all scales, the Falaco Soliton should be a natural artifact of both the sub-atomic and the cosmological worlds. The reason d'etre for this chapter is to bring the idea to the attention of other researchers who might find the concept of Falaco Solitons interesting and stimulating to their own research.

# Chapter 2
# COSMOLOGY AND THE NON EQUILIBRIUM VAN DER WAALS GAS

## 2.1 Cosmic Strings and Wheeler Wormholes

In the previous chapter is was mentioned that during the early stages of formation of the Falaco Solitons, caustics could be observed on the surface of density discontinuity that mimic the spiral arms so often observed in paper thin, almost "flat", galaxies such as our own Milky Way. The observation is enhanced if chalk dust (or dirt) is deposited on the surface of the pool during the first few seconds of Soliton formation. The observation is so stimulating that it leads to the conjecture that perhaps the spiral arm galaxies of the cosmos come in connected pairs. Could it be true that the Milky Way galaxy and its spiral arm companion, M31 are Falaco Solitons connected by some stabilizing thread? Only recently has photographic evidence appeared suggesting that galaxies may be connected by strings.


Indeed, the visual exhibition at the macroscopic level of dynamic topological coherent structures in a swimming pool, connected by a string, gives a level of credence to esoteric constructions of string theory. It is strange that the string theorists have ignored the experimental observations of Falaco Solitons where the "string" is neither microscopically small, nor folded into another dimension.

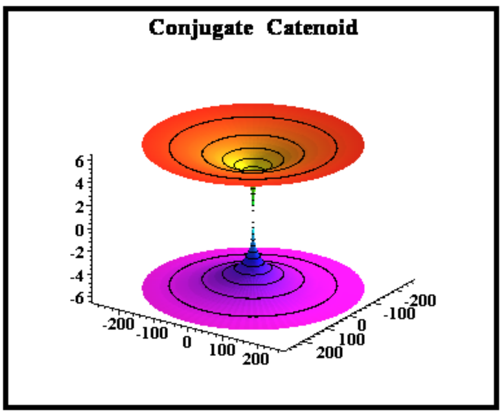
It is also extraordinary that the Falaco Solitons appear to be macroscopic realizations of a deformed Wheeler wormhole. The wormhole structure was presented early on by Wheeler (1955), but was considered to be unattainable in a practical sense. To quote Zeldovich p. 126 [267]

> "The throat or "wormhole" (in a Kruskal metric) as Wheeler calls it, connects regions of the same physical space which are extremely remote from each other. (Zeldovich then gives a sketch that topologically is equivalent to the Falaco Soliton). Such a topology implies the existence of 'truly geometrodynamic objects' which are unknown to physics. Wheeler suggests that such objects have a bearing on the nature of elementary particles and anti particles and the relationships between them. However, this idea

> has not yet borne fruit; and there are no macroscopic "geometrodynamic objects" in nature that we know of. Thus we shall not consider such a possibility."

This quotation dates back to the period 1967-1971. Now the experimental evidence (of Falaco Solitons) justifies Wheeler's intuition, which has so often been correct. Falaco Solitons are 'truly geometrodynamic objects'.

The minimal surface solution to the Navier - Stokes equations given in the preceding chapter suggests a connection between Falaco Soliton topological defect structures and Wheeler worm holes. The Falaco Soliton endcaps can be made to be compact by adding a closed component to the 1-form of Action. In particular, the additional component gives a circulation (without vorticity) that can be adjusted to compensate for the vorticity induced circulation (due to bulk rotation of the fluid) at a fixed radius, r.




The picture describes the minimal surface structure of a Falaco Soliton Pair.
By adding a singular vortex thread the topological structure can be made compact.

## 2.2 A Cosmological Conjecture based on Continuous Topological Evolution

The generation of almost flat spiral arm structures during the formation of the Falaco Solitons, and the idea that these structures could be macroscopic realizations of the Wheeler wormhole, suggests the possibility that topological defects are the fundamental causes of inhomogeneities in the night sky.

Recall, that size and shape have no crucial significance in a topological theory. Therefor, topological things and effects that appear at the microscopic and macroscopic scale should also appear at a cosmological scale. The objective of this chapter is to examine topological aspects and defects of thermodynamic physical systems - especially non-equilibrium thermodynamic systems - and their possible continuous topological evolution, creation, and destruction on a cosmological scale. There are two ways of utilizing the concepts of continuous topological evolution; both are based upon the assumption that the universe can be modeled in terms of an exterior differential 1-form of Action, $A$. The first method examines the features of the Pfaff topological dimension and its evolution producing long-lived topological defects and topologically coherent structures. The second method exploits the properties that the Jacobian matrix of the coefficients of the 1-form of Action creates a universal thermodynamic phase function, $\Theta$, in terms of the Cayley-Hamilton characteristic polynomial. That universal thermodynamic function contains a realization of the Universe as a universal, deformable van der Waals gas.

The cosmological creation and evolution of stars and galaxies will be interpreted herein in terms of a non equilibrium thermodynamic system of Pfaff Topological dimension 4 subjected to irreversible processes. As explained below, based upon this single assumption it is possible to devise a model of the universe which can be approximated in terms of the non equilibrium states of a very dilute van der Waals gas near its critical point. The stars and the galaxies are the topological defects, or coherent - but not equilibrium - structures of Pfaff topological dimension 3 formed by irreversible dissipative processes in this non equilibrium turbulent system of Pfaff topological dimension 4. The cosmology so constructed is opposite in viewpoint to those efforts which hope to describe the universe in terms of properties inherent in the quantum world of Bose-Einstein condensates, super conductors, and superfluids [246]. Both approaches utilize the ideas of topological defects, but thermodynamically the approaches are opposite. The quantum method involves, essentially, equilibrium systems, while the approach presented herein is based upon non equilibrium systems.

The topological theory of the ubiquitous van der Waals gas leads to a more mundane explanation of negative pressure (dark energy), string tension (dark matter), irreversible dissipation due to expansion ("volume viscosity"), and a Higgs potential (space time conformal inertial interaction). All of these concepts appear as natural consequences of a non equilibrium thermodynamics and a deformable version of a van der Waals gas near its critical point. These ideas arise without invoking quantum mechanics per se, and without assuming microscopic quantum vacuum fluctuations. Fluctuations are important, but in the sense that they are topological fluctuations about a guiding fiber of kinematic perfection. Perhaps of more importance is the fact that these thermodynamic consequences explicitly do not depend upon the geometric constraints of metric or connection, and indeed impose a different perspective on the concept of gravitational interaction as a possible topological effect, rather than a geometrical idea.

For example, the concept of a fixed point can be identified with a null eigenvector of a Jacobian matrix. The Jacobian matrix of a 1-form has both symmetric and anti-symmetric parts.

$$\mathbb{J}(\partial A_k/\partial x^m] = \mathbb{S} + \mathbb{A}. \tag{2.1}$$

The components of the anti-symmetric part can be put into correspondence with the exterior derivative of the 1-form of Action, $dA$. If the Pfaff topological dimension of $dA$ is 4, such that $K = dA\hat{}dA \neq 0$, then there does not exist a null eigen vector such that $i(V_{null})dA = 0$. Written in matrix language,

$$\text{"Electromagnetic" Work} \quad i(V_{null})dA = \mathbb{A} \circ |V_{null}\rangle \Rightarrow 0, \tag{2.2}$$

on a symplectic manifold of Pfaff topological dimension 4 cannot be true. However the concept of a fixed point implies that the similarity invariant of the fourth order Cayley Hamilton polynomial must vanish, $T_K = 0$. Hence:

$$\mathbb{J}\,|V_{null}\rangle = \mathbb{S}\,|V_{null}\rangle + \mathbb{A}\,|V_{null}\rangle = 0. \tag{2.3}$$

It follows that in such circumstances the symmetries must be balanced by the anti-symmetries in the sense that

$$\mathbb{S}\,|V_{null}\rangle = -\mathbb{A}\,|V_{null}\rangle\,. \tag{2.4}$$

The moral is that null vector constraints imply that symmetrical deformations $\mathbb{S}\,|V_{null}\rangle$ must be compensated by "electromagnetic" work $\mathbb{A}\,|V_{null}\rangle$ in the turbulent non-equilibrium domain of Pfaff topological dimension 4. The separation of "charge" is a possible remnant of "electromagnetic" work and occurs via irreversible processes in domains of Pfaff topological dimension 4.

If the domain is of Pfaff dimension 3, then there exist null eigen vectors of the 2-form $dA$ and symmetrical deformations are not necessarily linked to "electromagnetic" work.

### *2.2.1 Landau's argument for interactions of fluctuations*

The original motivation for the conjecture that the universe is a turbulent deformable van der Waals gas near its critical point is based on the classical theory of correlations of fluctuations presented in the Landau-Lifshitz volume on statistical mechanics [113]. When first reading Landau's ideas (about 1965), the present author made a written note in the textbook margin that Landau's ideas might be a method of explaining the fact that the night sky is not homogenous*, and instead is filled with objects (called stars) that appear to obey Newtonian gravitational attaction. However, the methods used in this chapter to describe cosmology of the universe are not statistical, not quantum mechanical, but instead are based on Cartan's methods of exterior differential forms and their application to the topology of thermodynamic systems and their continuous topological evolution (see Vol. 1, or [188]). Landau and Lifshitz emphasized that real thermodynamic substances, near the thermodynamic critical point, exhibit extraordinary large fluctuations of density and entropy. In fact, these authors demonstrate that for an almost perfect gas near the critical point, the correlations of the fluctuations can be interpreted as a 1/r potential giving a $1/\mathrm{r}^2$ force law of attraction. Hence, as a cosmological model, the almost, but not, perfect gas - such as a very dilute van der Waals gas - near the critical point yields a reason for both the apparent granularity of the night sky and for the $1/\mathrm{r}^2$ force law ascribed to gravitational forces between for massive aggregates. The stars are topological defects in the otherwise homogeneous cosmos. Landau also offers an argument for an inverse fourth power potential related to BE attraction or FD repulsion (p. 373 [113]). It is remarkable that the law of force is essentially the famous Maxwell $1/\mathrm{r}^5$ law for non-equilibrium gases (p 238 [89])

### *2.2.2 The Universe as a Turbulent van der Waals Gas near the Critical Point.*

A topological (and non statistical) thermodynamic approach can be used to demonstrate how a four dimensional variety can support a turbulent, non equilibrium, physical system with universal properties that are homeomorphic (deformable) to a van der Waals gas [208]. The method leads to the necessary conditions required for the existence, creation or destruction of topological defect structures in such a non equilibrium system. For those non-equilibrium physical systems that admit description in terms of an exterior differential 1-form of Action potentials of maximal rank, a Jacobian

*Counter to the then classic theory of a homogeneous cosmological universe matrix can be generated in terms of the partial derivatives of the coefficient functions that define the 1-form of Action. When expressed in terms of intrinsic variables, known as the similarity invariants, the Cayley-Hamilton 4 dimensional characteristic polynomial of the Jacobian matrix generates a universal phase equation as a 4th order polynomial in the (complex) eigen functions of the matrix. Certain topological defect structures can be put into correspondence with constraints placed upon those (curvature) similarity invariants generated by the Cayley-Hamilton 4 dimensional characteristic polynomial. These constraints define equivalence classes of topological properties. It is assumed that the universe can be represented by such a 1-form of Action of Pfaff Topological dimension 4.

The characteristic polynomial, or Phase function, can be viewed as representing a family of implicit hypersurfaces. The hypersurface has an envelope which is related to a swallowtail bifurcation set of dynamical system theory when the hypersurface is constrained such that the linear similarity invariant vanishes (this constraint corresponds to the idea that the trace of the Jacobian matrix vanishes). The swallowtail defect structure is homeomorphic (can be deformed) to the Gibbs surface of a van der Waals gas.

Another possible defect structure corresponds to the implicit hypersurface surface constrained such that the quartic similarity invariant vanishes (this constraint corresponds to the idea that the determinant of the Jacobian matrix vanishes). The constraint implies that at least one eigenvalue is zero. On 4 dimensional variety (space-time) , this non degenerate hypersurface constraint leads to a cubic polynomial that always can be put into correspondence with a set of non equilibrium thermodynamic states whose kernel represents the equation of state of a van der Waals gas.

Hence the universal topological method for creating a universal phase function in terms of the Cayley-Hamilton theorem for the Jacobian matrix of a 1-form of Action, leads to a thermodynamic system that can be deformed into a van der Waals gas. Near the critical point, a low density turbulent non equilibrium media leads to the setting examined statistically by Landau and Lifshitz in terms of classical fluctuations about the critical point.

The conjecture presented herein is that non equilibrium topological defects in a non equilibrium 4 dimensional medium represent the stars and galaxies, which are gravitationally attracted singularities (correlations of fluctuations of density fluctuations) of a real gas near its critical point. Note that the Cartan methods (in contrast to metrical theories) do not impose (*a priori)* a constraint of a metric, connection, or gauge, but do utilize the topological properties associated with constraints placed on the similarity invariants of the universal phase function.

### *2.2.3 Results*

Based upon the single assumption that the universe is a non-equilibrium thermodynamic system of Pfaff topological dimension 4 leads to a cosmology where the universe, at present, can be approximated in terms of the non-equilibrium states of a very dilute van der Waals gas near its critical point. The stars and the galaxies are the topological defects and coherent (but not equilibrium) self-organizing structures of Pfaff topological dimension 3 formed by irreversible topological evolution in this non-equilibrium system of Pfaff topological dimension 4.

The turbulent non-equilibrium thermodynamic cosmology of a real gas near its critical point yields an explanation for:

1. The granularity of the night sky as exhibited by stars and galaxies.

2. The Newtonian law of gravitational attraction proportional to $1/r^2$ [113].

3. The conformal expansion of the universe as an irreversible phenomenon associated with Quartic similarity invariants in the thermodynamic phase function, and conformally related to dissipative effects [169].

4. The possibility of domains of negative pressure (explaining what has recently been called dark energy) due to a *classical* "Higgs" mechanism for aggregates below the critical temperature (Cubic similarity invariants or 3rd order curvature effects).

5. The possibility of domains where gravitational effects (quadratic similarity invariants, or 2nd order Gauss curvature effects) appear to be related to entropy and temperature properties of the thermodynamic system.

6. The possibility of cohesion properties (explaining what has recently been called dark matter) due to string or surface tension (linear similarity invariants or 1st order Mean curvature effects).

7. Black Holes (generated by Petrov Type D solutions in gravitational theory [42]) are to be related to Minimal Surface solutions to the Universal thermodynamic 4th order Phase function.

In this chapter, a review of the thermodynamic properties of the Phase function will be made for the van der Waals gas. Then it will be demonstrated that the results for the van der Waals gas are deformable representations for any thermodynamic system that can be encoded by a 1-form of Action on a symplectic domain of Pfaff topological dimension 4. The theory is not complete, but a number of conjectures that offer explanations of current cosmology are made. These ideas do not depend a priori upon metric or connection of current gravity theories, nor do the depend upon the zoo of quantum virtual particles and quanta that currently are used to describe the cosmological vacuum,

## 2.3 The Ubiquitous Universal van der Waals Gas

In this section, the algebraic and thermodynamic features of both an equilibrium and non-equilibrium van der Waals gas are reviewed. Then, the general universal method for generating the thermodynamic phase function for a non-equilibrium thermodynamic system of Pfaff Topological dimension 4 is carried out in some detail. The concept and physical importance of the Pfaff Topological dimension is summarized in this chapter, but is expressed in much more detail in Vol. 1.

The simplistic equation of state for an ideal (*perfect*) gas consisting of "parts" that do not interact is given by the equation,

$$\text{Ideal Gas:} \quad P/RT = \rho = n/V. \tag{2.5}$$

The "perfect" gas does not encode certain thermodynamic features (phase transitions and critical point behavior) which are observable in "real" gases. It has been argued that the "real" gas consists of $n$ geometric "parts" that interact with one another, in contrast to an ideal gas, where geometric features (size and shape) of such "parts" and their interactions had been ignored. The classic interpretation is that $n$ represents the number of moles, where moles is interpreted in terms of microscopic "molecules". Motivated by such ideas, van der Waals created, phenomenologically, an equation of state for "real" gases in terms of two parameters, $a$ and $b$, which were introduced to encode the interaction and geometric size features of the "molecular" components. The resulting formula for an equation of state was cubic in the molar density, $\rho = n/V$.

$$\text{Van der Waals:} \qquad P = \frac{\rho RT}{1 - b\rho} - a\rho^2, \tag{2.6}$$

$$\text{or:} \qquad ab\rho^3 - a\rho^2 + \{RT + bP\}\rho - P = 0. \tag{2.7}$$

The formula has enjoyed remarkable success for qualitatively explaining the thermodynamic features of real gases. The formula represents an implicit surface in the space of variables, $\{P, T, \rho\}$. However, the development was phenomenological, and although motivated by the concept of microscopic "molecules", the fundamental properties were independent from the geometric size of its parts. From a topological point of view, size is not of primary concern. Topological properties of interest to this monograph are independent from size and shape. What is important is the number of parts, the number of holes, the limit points, the orientation, and other homeomorphic properties that can change under continuous topological evolution. As Sommerfeld has said (without explicit reference to topology, but inferring that microscopic molecules are not of thermodynamic importance):

> "The atomistic, microscopic point of view is alien to thermodynamics. Consequently, as suggested by Ostwald, it is better to use moles rather than molecules." p. 11 [224].

The ideal gas approximation has been found to be of utility to the study of agglomerates of parts that range from the geometric size of nuclei to the geometric size of stars. A major purpose of this section is to demonstrate the universality of the topological van der Waals gas to the study of condensates of all types of "parts" in non equilibrium configurations based upon topological issues.

By differentiating the Van der Waals equation of state with respect to the molar density, it can be determined that there exists a "critical point" on the hypersurface at which the values of the Pressure, Temperature and molar density take on values such that

$$\text{at the critical point, } P_c/T_c\rho_c = \text{ constant.} \tag{2.8}$$

When the thermodynamic variables are expressed in terms of dimensionless (reduced) variables, scaled in terms of their values at the critical point, the values of those parameters, $a$ and $b$, which were used to model the geometric - interaction - shape and geometric size features cancel out. In this sense, the "renormalized" or "reduced" van der Waals equation of state becomes an element of topological equivalent class with universal topological properties (independent from scales). The reduced van der Waals equation is independent from the size and interaction magnitudes of its component parts. In terms of these dimensionless variables,

$$\widetilde{P} = P/P_c,\ \widetilde{V} = V/V_c,\ \widetilde{T} = T/T_c,\ \rho = n/V, \tag{2.9}$$

the classic van der Waals equation may be considered as a cubic constraint on the space of variables $\{n; \widetilde{P}, \widetilde{T}, \widetilde{\rho}\}$ where $\widetilde{\rho} = n/\widetilde{V}$ is defined as the dimensionless molar density. The reduced universal van der Waals equation of state is given by the classic cubic expression,

$$\begin{aligned} & \text{Classic Van der Waals equation} \\ & \text{(in reduced variables)} \\ 0 &= \widetilde{\rho}^3 - 3\widetilde{\rho}^2 + \{(8\widetilde{T} + \widetilde{P})/3\}\widetilde{\rho} - \widetilde{P}. \end{aligned} \tag{2.10}$$

It is this scale independent polynomial equation that promotes a correspondence between topological ideas and thermodynamics. This formula should be memorized, for it yields a direct connection of the van der Waals gas and a cubic polynomial.

In the development that follows the formula will be related to the Cayley-Hamilton equation for a non degenerate 4x4 real matrix. The Cayley-Hamilton formula is of the type

$$\text{Cayley-Hamilton polynomial} = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi + T_K = 0. \tag{2.11}$$

The eigenvalues of the real matrix can be complex numbers, but as the similarity coefficients, $\{X_M, Y_G, Z_A, T_K\}$ are all real, classic analysis yields the result that eigenvalues form three equivalence classes:

1. 4 real eigenvalues.

2. 2 real eigenvalues, and 1 complex eigenvalue and its 1 complex conjugate.

3. 2 complex eigenvalues, and their 2 complex conjugates.

If $T_K = 0$, then the Cayley-Hamiltonian equation becomes,

$$\text{Cayley-Hamilton polynomial} = (\xi^3 - X_M\xi^2 + Y_G\xi^1 - Z_A)\xi = 0, \tag{2.12}$$

and the similarity coefficients are related to the "Curvatures" of the implicit surface induced by the molar density. The first (cubic) factor can be put into direct correspondence with the Classic van der Waals equation

$$\text{Van der Waals } \xi = \widetilde{\rho}, \tag{2.13}$$

$$\text{Linear} :$$

$$\xi_1+\xi_2+\xi_3+\xi_4 = X_M \Rightarrow 3, \tag{2.14}$$

$$\text{Quadratic} :$$

$$\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1+\xi_1\xi_4+\xi_2\xi_4+\xi_3\xi_4 = Y_G \Rightarrow (8\widetilde{T}+\widetilde{P})/3 \tag{2.15}$$

$$\text{Cubic} :$$

$$\xi_1\xi_2\xi_3+\xi_1\xi_2\xi_4+\xi_2\xi_3\xi_4+\xi_3\xi_1\xi_4 = Z_A \Rightarrow \widetilde{P} \tag{2.16}$$

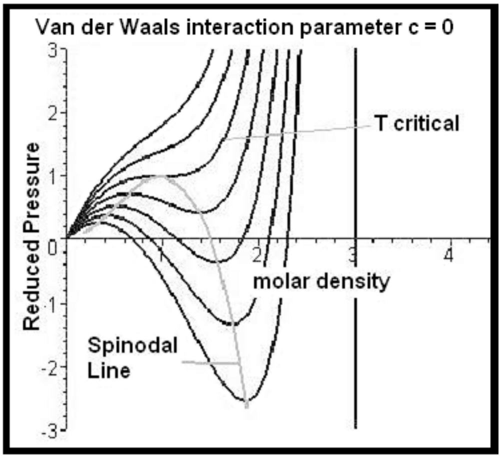
$$\text{Quartic} :$$

$$\xi_1\xi_2\xi_3\xi_4 = T_K. \tag{2.17}$$

In the above formulas the $\xi_1, \xi_2, \xi_3, \xi_4$ are the local eigenvalues of the Jacobian matrix.

Forces and energies associated with the Linear curvature are typical of surface tension effects. It becomes apparent that forces and energies associated with the Cubic similarity invariant represent the Pressures of interactions. The Gauss quadratic similarity invariant is dominated by temperature, with a pressure contribution.

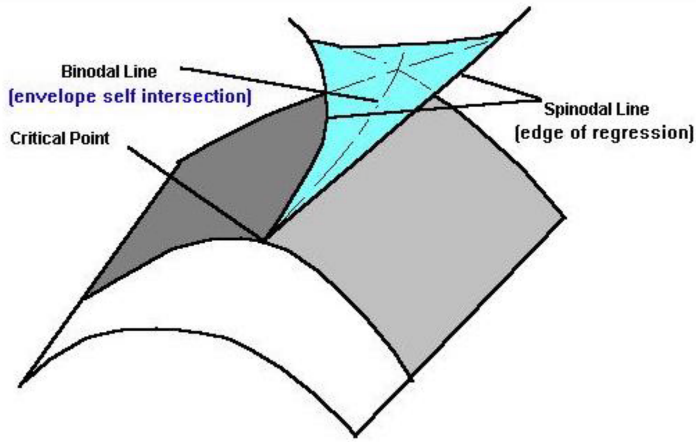
A $\widetilde{P}, \widetilde{\rho}$ projection of the implicit universal van der Waals surface is given in the next figure. The diagram displays a critical isotherm that separates a single phase (the gas) from the different topological domains that can be interpreted as liquids and vapor. The shape of the critical isotherm should be remembered, for above the critical isotherm, there exists a unique value for the pressure, and below the critical isotherm there is more than one value for the pressure. This feature represents a topological property of the van der Waals gas, and will have importance in the study of non equilibrium systems. Of interest to cosmologists, the Pressure for the van der Waals gas, for values below the critical isotherm, can take on negative values. As will be shown below, the Phase function below the critical isotherm has the shape of a Higgs (quartic) potential.




Note the regions of Negative Pressure

There exists a dual surface to the equation of state as defined by a Legendre transformation to the Gibbs function, $g = u - Ts + Pv$. The implicit surface defined by the Gibbs function (for a van der Waals gas) is not single valued, and appears as a deformation of a swallow tail bifurcation set. The actual Gibbs surface for the van der Waals gas can be numerically computed and is presented in the next Figure. An accurate drawing of the 3D Gibbs surface appears only occasionally in thermodynamic text books. Most presentations, if in 3D, are given by sketches, and not by actual computations. For example, in [2] p.196, the Gibbs surface misses the fact that the Spinodal line forms a cusp at the critical point. In the figure below, the salient features are displayed by numeric computation of the Gibbs surface for the van der Waals gas. Remarkably, the dual Gibbs surface displays the envelope features of the Universal Phase function. Recall that the envelope is an element of the Renormalization Group[108]. The cuspoidal critical point singularity, the winged cusp representing the Spinodal line, and the Binodal self intersection are universal topological features of the discriminant (envelope) hypersurface. In the figure below, the white region is where the temperature is above the critical isotherm and represents the pure gas. The other sectors are below the critical isotherm, and are influenced by the "Higgs" features of the Phase potential. The dark gray sector represents the fluid phase, and the light gray sector represents the vapor phase. The light "blue" sector represents the unstable mixed phase region.

**Conjecture** *The topological features of the van der Waals gas* *are universal features (deformation invariants for all physical systems that admit a realization over 4D space-time variety. The van der Walls gas is one element of a topological equivalence class.*




**A Universal Thermodynamic Swallowtail**
The Gibbs surface of a Van der Waals gas

Historically the two implicit surfaces defined by the reduced van der Waals equation became quite useful to chemical engineers and led to the law of corresponding states. If properties of a gas near its critical point could be measured, then the law of corresponding states permitted estimates to be made for the properties of the gas by comparison to the universal van der Waals model. The topological results were independent of the geometric parameters of size, $b$, and interaction, $a$. In this and following sections, the universal topological features of the Phase function and the Gibbs surface of the generalized† van der Waals gas will be developed and applied to non equilibrium systems. The "generalization" consists of adding a contribution to the reciprocal volume use in the interaction term. Recall that non equilibrium requires that the Pfaff topological dimension of the Action 1-form is 3 or greater in certain regions. Non equilibrium systems

†The "generalization" consists of adding a contribution to the reciprocal volume use in the interaction term.

can exist in "stationary" states where there topological coherence properties are evolutionary invariants.

The principle (universal) topological defect structure of a van der Waals gas is the existence of a critical *point*. When expressed in terms of reduced coordinates, $\{\widetilde{P}, \widetilde{T}, \widetilde{\rho}\}$, the critical point of the implicit surface representing the equation of state, is where the reduced (dimensionless) functions all have the common value unity. The topological significance of the critical isotherm, which passes through the critical point, has already been mentioned above.

Another important topological defect structure is the existence of a Spinodal *line*, of ultimate phase stability, consisting of two parts that meet in a cusp at the critical point. The Spinodal line will be established by an edge of regression in the Gibbs surface.

Yet another topological defect structure is exhibited by the Binodal line, defining portions of a ruled *surface* representing the region of mixed phases. The Binodal line can be described by a deformation of a pitchfork bifurcation emanating from the critical point, and line which outlines the domain of mixed phases. The domain of mixed phase is related to regions where the Pfaff topological dimension of the encoded physical system (the 1-form of Action) is at least 3. The domains of isolated single phase are related to regions where the Pfaff topological dimension is 2 or less.

A lot can be learned from the van der Waals example, for its features are experimentally verifiable. The universal qualities are obtained in terms of variables that represent deformations and non equilibrium extensions of the van der Waals properties. The van der Waals internal energy is a Lagrangian (phase) function in terms of extensive variables. In the language of classical mechanics, the Lagrange function is a function of the base variables, $q^k$, and their first derivatives, $v^k$, or velocity "extensive" functions. A Legendre transformation leads to a Hamiltonian function in terms of intensive variables, the momenta, $p_k$. The classic van der Waals phase function defines a hypersurface in the space of extensive variables of entropy, $S$, volume, $V$, and energy, $U$. A Legendre transformation produces a "Gibbs-Hamiltonian" function of intensive variables, temperature $T$, pressure, $P$, and Gibbs free energy, $Gibbs$.

The zero sets of certain algebraic combinations of the similarity curvature invariants of these hypersurfaces define universal topological features of the physical system, which are of value to the study of both equilibrium and non equilibrium systems. Rather that formulating the non equilibrium universal phase equation in a phenomenological manner, it will be demonstrated that such a universal phase function can be generated as the Cayley-Hamilton polynomial equation of the Jacobian matrix for the 1-form of Action, $A$, that represents the physical system. The topological Pfaff dimension of $A$ permits the delineation between those phase functions that represent non equilibrium systems and those that do not.

The following subsections first will discuss the ideas associated with Extensive and Intensive variables. Then the classic van der Waals expression for a Phase equation will be used to define an internal energy surface in terms of intensive variables. A dual construction will be used to create the Gibbs energy in terms of intensive variables. The Gibbs surface is deformably (topologically) equivalent to the swallow-tail discriminant, or envelope of the classic phase equation. After this review of classical theory in the language of topological evolution, the theory will be extended to include non equilibrium systems of the closed and open types.

### *2.3.1 The Phase function for a van der Waals Gas*

In the classical development of thermodynamics, the van der Waals gas is often used as a cornerstone example. However, the phase function, $\Theta$, given in many textbook treatments is not explicitly homogeneous of degree 1 in the extensive variables. A homogeneously correct formulation, to within a constant, is given by the relation:

$$\Theta\{...S,V,n;U\} = n[e^{\frac{S}{nC_v}}(\frac{V}{n}-b)^{-\frac{R}{C_v}} - \frac{a}{(\frac{V}{n}+cb)} - \frac{U}{n}] \Rightarrow 0. \tag{2.18}$$

The constant $b$ is a representative size of the "particles" that make up molar quantities of the gas. Currently, it is usual to consider the "molar" quantities to be microscopic molecules, but the molar quantities from a topological perspective can be any size, ranging from nuclei to stars. To repeat Sommerfeld's statement:

> "The atomistic, microscopic point of view is alien to thermodynamics. Consequently, as suggested by Ostwald, it is better to use moles rather than molecules." p. 11 [224].

The constant $a$ is representative of the interaction forces between the molar quantities. The term $a/(\frac{V}{n})^2$ has been described by Sommerfeld as representing the "forces (or energy) of cohesion" p. 58 [224] . Note that a correction factor, $cb$, has been added to the historical collision term $a/(V/n) \to a/(V/n+cb)$ in order to account for the finite interaction size (or an effective scattering wavelength, or coherence length, $cb$) of the interacting molar particles. The coefficient $c$ can be adjusted to give a better fit of the van der Waals gas equation to the experimental data of $\Omega_c = (nRT_c/P_cV_c)$ at the critical point.

This equation for $\Theta\{S, V, n; U\}$ satisfies the Euler condition for homogeneity of degree 1, with respect to the *extensive* variables $\{S, V, n; U\}$ :

$$U\partial\Theta/\partial U + V\partial\Theta/\partial V + S\partial\Theta/\partial S + n\partial\Theta/\partial n - \Theta = 0. \tag{2.19}$$

The partial derivatives the phase function, $\Theta$, with respect to the extensive variables may be used to define *intensive* variables, $(P, T, \mu, \beta)$,

$$(P = -\partial\Theta/\partial V,\ T = \partial\Theta/\partial S,\ \mu = -\partial\Theta/\partial n, \beta = -\partial\Theta/\partial U). \tag{2.20}$$

From the phase function ( 2.18 ), partial differentiation yields:

$$T = \frac{\partial}{\partial S}(\Theta) = (e^{\frac{S}{nC_v}}(\frac{V}{n} - b)^{-\frac{R}{C_v}})/C_v, \tag{2.21}$$

$$P = -\frac{\partial}{\partial V}(\Theta) = \frac{nRT}{V - bn} - a\frac{n^2}{(V + cbn)^2}. \tag{2.22}$$

Differentiating $P$ with with respect to $V$ yields

$$\partial P/\partial V = -\frac{nRT}{(-V + bn)^2} + 2a\frac{n^2}{(V + cbn)^3}, \tag{2.23}$$

and differentiation again leads to

$$\partial^2 P/\partial V^2 = -2\frac{nRT}{(-V + bn)^3} - 6a\frac{n^2}{(V + cbn)^4}. \tag{2.24}$$

The classic argument to determine the critical point sets these differential relations to zero. The values of the thermodynamic variables at the critical point are:

$$V_c = bn(2c + 3), \qquad T_c = \frac{8a/27}{bR(c + 1)}, \quad P_c = \frac{a/27}{b^2(c + 1)^2}. \tag{2.25}$$

Note that if the critical molar density is defined as $\rho_c = n/V_c$, the previous equations which leads to the universal constant, $\Omega_c$, independent from the geometrical parameters $\{a, b\}$:

$$\Omega_c = R(\rho_c T_c/P_c) = nRT_c/P_cV_c = 8\frac{c + 1}{2c + 3}. \tag{2.26}$$

The reciprocal of $\Omega_c$ is often defined as the compressibility, $Z = 1/\Omega_c$. For the van der Waals gas ($c = 0$), $\Omega_c$=1/.375, but for many real gases, the experimental value is closer to $\Omega_c$=1/.27. This result is in good agreement with the value of $c = 4$.

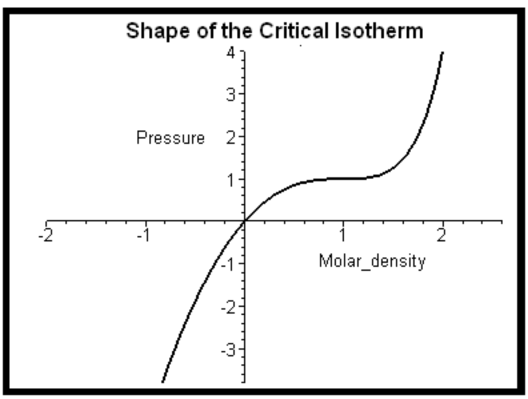
For the classic van der Waals gas ($c = 0$), a rescaled equation of state can be obtained in terms of the dimensionless variables, scaled by their values at the critical point.

$$\widetilde{\rho} = \rho/\rho_c \quad \widetilde{T} = T/T_c \quad \widetilde{P} = P/P_c \tag{2.27}$$

$$0 = \widetilde{\rho}^3 - 3\widetilde{\rho}^2 + \{(8\widetilde{T} + \widetilde{P})/3\}\widetilde{\rho} - \widetilde{P}. \tag{2.28}$$

At the critical point, $\widetilde{\rho} = 1$, $\widetilde{T} = 1$, $\widetilde{P} = 1$. What is remarkable is that the coefficients $a$ and $b$ introduced to better account for the properties of the "particles" cancel out in the rescaled formulas. It is this feature that makes the van der Waals gas formulas have a universal appeal, and leads to the idea of "corresponding states".

The classic rescaled van der Waals formula leads to a critical isotherm that topologically separates the pure gas phase from those regions that admit liquid, or vapor, coexistent mixed phases. The "universal shape of the critical isotherm is given in the figure below. It is a topological invariant and is to be recognized by its distinctive shape.




For arbitrary coefficient $c$, the cubic formula for the reduced equation of state is also independent from the van der Waals parameters a and b, but is of a somewhat more complicated format:

$$\begin{aligned}\Theta &= (8\widetilde{T}c^3 + (27 + 8\widetilde{T} + \widetilde{P})c^2 + 54c + 27)\rho_c^3 \\ &\quad +((-27 - \widetilde{P} + 16\widetilde{T})c^2 + (-54 + 16\widetilde{T} + 2\widetilde{P})c - 27)\rho_c^2 \\ &\quad +((-2\widetilde{P} + 8\widetilde{T})c + 8\widetilde{T} + \widetilde{P})\rho_c) - \widetilde{P} \Rightarrow 0. \qquad (2.29) \\ \rho_c &= \widetilde{\rho}/(2c+3) \qquad (2.30)\end{aligned}$$

### *2.3.2 The Jacobian Matrix of the Action 1-form.*

The Cartan topological methods of exterior differential forms emphasize the antisymmetric features of a physical system, especially through the antisymmetric matrix that encodes the 2-form $dA$. The more geometric formulation of the van der Waals gas, as described in the previous section, can also be obtained from the symmetrical differential properties of the coefficients of 1-form of Action, $A$. It will be assumed that the 1-form of Action, $A$, that encodes the physical system, is of Pfaff topological dimension 4, except on certain subspaces of the 4 independent variables.

It should be noted that from the point of view of dimensional analysis each term in the Action 1-form is presumed to of the same "physical" dimension. The coefficients are conjugate to the differentials. For projective realizations, the next step assumes that the coefficients are all of the same physical dimension, and the differentials are all of the same "physical" dimension. This latter assumption is stronger than the idea that the coefficients are intensive and the differentials are extensive.

The idea to be exploited in that which follows is that the Jacobian matrix $\mathbb{J} = \left[\partial A_k/\partial x^j\right]$ of partial derivative functions, created from the coefficients of the 1-form of Action, satisfies a Cayley-Hamilton matrix polynomial equation, and a complex algebraic polynomial equation in terms of the eigenvalues, $\xi$, of the Jacobian matrix.

$$\text{Cayley Hamilton polynomial} = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi^1 + T_K = 0. \quad (2.31)$$

The coefficients of the polynomials $\{X_M, Y_G, Z_A, T_K\}$ are invariant with respect to similarity transformation of the Jacobian matrix, and in this (restricted) sense the method is universal. The Jacobian matrix contains both symmetric and anti-symmetric components, where the 2-form, $dA$, emphasizes the anti-symmetric features of the partial derivatives of the 1-form coefficients. The symmetric similarity properties are more related to euclidean geometric properties of the physical system, but it should be realized that congruence (size) and distance are additional requirements necessary for a euclidean geometry. Similarity transformations are special projective transformations that preserve parallelism and orthogonality (or better said, preserve points at infinity and the special point that defines the origin.)

It is assumed that this characteristic equation, as a polynomial of 4th degree, is in effect a Universal Thermodynamic Phase function, $\Theta(x, y, z, t;\ \xi)$ :

$$\text{Cayley-Hamilton polynomial} = \Theta(x, y, z, t;\ \xi) = 0. \tag{2.32}$$

The Phase function is distinct for different categories of coefficient functions that make up the 1-form of Action, $A$, but all such Phase functions are related to the deformation equivalence classes that include the classic van der Waals gas. The Universal Phase function defines a family of implicit hypersurfaces in the space of "universal" coordinates defined in terms of the similarity invariants, $\{X_M, Y_G, Z_A, T_K\}$. It will be demonstrated how and when the similarity invariants can be related to "curvatures" of the universal implicit hypersurface. However, no metric is used explicitly to define the "curvatures".

The non equilibrium extensions of the van der Waals gas (of Pfaff topological dimension 4) are to a certain extent encoded in the third and fourth order similarity invariants, $Z_A$ and $T_K$, and the possibilities that the polynomial can have complex roots. In order to describe topologically isolated or equilibrium systems it is necessary (but not sufficient) these third and fourth order similarity invariants vanish. The similarity invariants are in effect symmetric averages of eigen values, which ignore the possible system antisymmetries. It is this difference the characterizes the failure of geometric concepts, (the quadratic metric form) and theories built on such symmetric constraints, to capture thermodynamic irreversibility.

In the special isolated-equilibrium cases, the topological features of a universal thermodynamic critical point, and a Spinodal line of ultimate phase stability have realizations in terms of topological constraints on the phase function implicit hypersurface that represents the universal equilibrium van der Waals gas. When written in terms of curvatures it can be demonstrated that the zero set of the quadratic similarity invariant (the Gauss curvature) represents the Spinodal line, or the edge of regression in the Gibbs surface, of a van der Waals gas. The thermodynamic critical point occurs when both the Mean curvature and the Gauss curvature of the equilibrium surface vanish. It is this universality that gives credence to the idea that the 4 dimensional universe could be represented as a non equilibrium van der Waals gas near its critical point [208]. These concepts will be extended to the non equilibrium systems in that which follows.

### *2.3.3 The Non Equilibrium Characteristic Phase Function*

The 1-form of Action, used to encode a physical system, contains other useful topological information, as well as geometric information. Reconsider the details of an open thermodynamic system generated by a 1-form of Action, $A$, of Pfaff topological dimension 4. The component functions of the Action 1-form can be used to construct a 4x4 Jacobian matrix of partial derivatives, $[\mathbb{J}_{jk}] = [\partial(A)_j/\partial x^k]$. In general, this Jacobian matrix will be a 4 x 4 matrix that satisfies a 4th order Cayley-Hamilton characteristic polynomial equation, $\Theta(x, y, z, t;\ \xi) = 0$, with 4 perhaps complex roots representing the 4 perhaps complex eigenvalues, $\xi_k$, of the Jacobian matrix.

$$\Theta(x, y, z, t;\ \xi) = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi^1 + T_K \Rightarrow 0. \tag{2.33}$$

The Cayley-Hamilton polynomial equation defines a family of implicit functions, $X_M, Y_G, Z_A, T_K$, in the space of real variables, $(x, y, z, t)$. The functions $X_M, Y_G, Z_A, T_K$ are the real similarity invariants of the Jacobian matrix, even though the eigenvalues may be complex. If the eigenvalues are distinct, then the similarity invariants are given by the expressions:

$$X_M = \xi_1 + \xi_2 + \xi_3 + \xi_4 = Trace\,[\mathbb{J}_{jk}]\,, \tag{2.34}$$

$$Y_G = \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1 + \xi_4\xi_1 + \xi_4\xi_2 + \xi_4\xi_3, \tag{2.35}$$

$$Z_A = \xi_1\xi_2\xi_3 + \xi_4\xi_1\xi_2 + \xi_4\xi_2\xi_3 + \xi_4\xi_3\xi_1, \tag{2.36}$$

$$T_K = \xi_1\xi_2\xi_3\xi_4 = \det\,[\mathbb{J}_{jk}]\,. \tag{2.37}$$

The 4 fold degeneracy that defines the critical point has roots $\xi = [1, 1, 1, 1]$, such that $X_M = 4,\ Y_G = 6,\ Z_A = 4,\ T_K = 1$. The critical point is not a fixed point (with zero eigenvalues). These results facilitate the law of corresponding states.

It should be noted that the characteristic polynomial is constructed from the "symmetric" properties of the Jacobian matrix, in the sense that the similarity coefficients are real combinations of complex numbers. On the other hand the anti-symmetric components of the Jacobian matrix are emphasized by the Pfaff topological dimension constructed from anti-symmetric differential forms, whose coefficients are the components of the anti-symmetric parts of the Jacobian matrix. For example, there exist 1-forms of Action that are of Pfaff topological dimension 4, yet the 4th order symmetric similarity invariant $T_K \Rightarrow 0$. Similarly, there are examples where $Z_A$ and $T_K$ both go to zero, but the system defined by the 1-form of Action, $A$, is of Pfaff dimension 3, and therefor defines a non equilibrium system. If the Quartic term $T_K$

vanishes, then there exists a null eigen vector for the Jacobian matrix. If the Pfaff topological dimension of the 1-form $A$ is equal to 4, then there cannot exist a null eigenvector for the anti-symmetric part of the Jacobian matrix. The null eigenvector for the Jacobian must have non-zero results for both the symmetric and anti-symmetric parts of the Jacobian matrix, separately. In particular, the 1-form of Work, $W = i(V_\xi)dA$ evaluated for the null eigen vector of the Jacobian matrix, can not be zero. The effect of the null eigen vector on the symmetric parts of the Jacobian must cancel the effects of the 1-form of Work. Hence, as noted above for symplectic systems:

$$\mathbb{J}\left|V_{null}\right\rangle = \mathbb{S}\left|V_{null}\right\rangle + \mathbb{A}\left|V_{null}\right\rangle = 0. \tag{2.38}$$

It follows that in such circumstances the symmetries must be balanced by the anti-symmetries in the sense that

$$\mathbb{S}\left|V_{null}\right\rangle = -\mathbb{A}\left|V_{null}\right\rangle. \tag{2.39}$$

The moral is that null vector constraints imply that symmetrical deformations $\mathbb{S}\left|V_{null}\right\rangle$ must be compensated by "electromagnetic" work $\mathbb{A}\left|V_{null}\right\rangle$ in the turbulent non-equilibrium domain of Pfaff topological dimension 4. The separation of "charge" is a possible remnant of "electromagnetic" work and occurs via irreversible processes in domains of Pfaff topological dimension 4.

From the theory of strings and surface tension, the $X_M$ term is - in a sense - a linear deformation contribution to the "energy" of the system. The coefficient $Y_G$ can be related to the Gauss (quadratic) curvature of the system, and is related to an area deformation contribution. The coefficient $Z_A$ can be related to the Interaction (Cubic) curvature of the system, and is related to a volume deformation contribution (a Pressure) to the "energy". The last term $T_K$ is a quartic contribution and can be related to an expansion or contraction of the 4 dimensional volume element.

Symbolically, multiply the phase function by $u/\xi^4$ and and consider $u/\xi$ to be a length deformation, $\delta_{Length}$, $u/\xi^2$ to be an area deformation, $\delta A_{rea}$, $u/\xi^3$ to be a volume deformation, $\delta_{Vol}$, and $u/\xi^4$ to be an space-time expansion deformation, $\delta_{Exp\_xyzt}$. The suggestive formula becomes

$$\Theta = u - X_M \cdot \delta_{Length} + Y_G \cdot \delta_{Area} \tag{2.40}$$

$$-Z_A \cdot \delta_{Vol} + T_K \cdot \delta_{Exp\_xyzt} \tag{2.41}$$

and by comparison with a van der Waals gas,

$$X_M \approx \text{"String or Surface\_tension"} \tag{2.42}$$

$$Y_G \approx \text{"Temperature - Entropy"} \tag{2.43}$$

$$Z_A \approx \text{"Pressure - Interaction"} \tag{2.44}$$

$$T_K \approx \text{xyzt- "Higgs" Expansion - Rotation} \tag{2.45}$$

Automatically, the phase function incorporates string or surface tension effects through $X_M$, where $X_M$ can be related to a mean four dimensional curvature expression. Gravity effects, related to the 4D Gauss curvature, $G = Y_G/6$ are "area" related. From the idea that the entropy of a gravitational black hole is related to an area, and the fact that the phase formula for a van der Waals gas implies that $Y_G$ is dominated by the temperature (see eq(2.13), the universal phase formula suggests that the idea of gravity (and the Gauss curvature) is a temperature - entropic concept, contributing energy of the type $TS$. The phase formula for a van der Waals gas implies that the $Z_A$ coefficient is related to Pressure (which can be both negative or positive), and the energy contribution is of the type $PV$. The last term represents a 4D xyzt expansion, which from the topological theory of thermodynamics presented above can be related to irreversible dissipation.

It is sometimes more convenient to express the similarity invariants in terms of their averages, where the average is determined by dividing by the number of non zero eigenvalues. This leads to a sequence of maps from the original variety of independent variables, $\{x, y, z, t\} \Rightarrow \{X_M, Y_G, Z_A, T_K\} \Rightarrow \{M, G, A^\star, K\}$. (Note that the symbol $A^\star$ is used for the Adjoint cubic average in order to eliminate confusion with the 1-form $A$.) When the averaged similarity invariants are treated as generalized coordinates, then the characteristic polynomial becomes a Universal Phase function, and will be used to encode universal thermodynamic properties.

A similar procedure can be applied to domains of lesser dimension. For example, suppose the dimension of the domain is reduced from Pfaff dimension 4 to Pfaff dimension 3 by the constraint that the determinant $T_K$ vanishes (this corresponds to the reduction of a non equilibrium turbulent system to a non equilibrium non turbulent system which can support steady states). The Phase equation must have one null eigen value, that represents a null eigenvector, or fixed point of the Jacobian matrix. The Phase equation with one eigenvalue = to zero (say $\xi_4 = 0$) reduces to

$$\Theta(x,y,z,t;\ \xi) = (\xi^3 - X_M\xi^2 + Y_G\xi^1 - Z_A)\xi \Rightarrow 0, \quad (2.46)$$

$$\text{with } X_M = (\xi_1 + \xi_2 + \xi_3), \quad (2.47)$$

$$Y_G = (\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1), \quad (2.48)$$

$$Z_A = \xi_1\xi_2\xi_3 = Z_A, \quad (2.49)$$

> **Conjecture** *An objective herein is to exploit the striking similarity between the cubic factor of the 3D phase equation (eq. (2.46)), and the cubic equation of the rescaled van der Waals gas given by equation (2.28 ). The fundamental assumption is that the eigen value of the Cayley-Hamilton characteristic polynomial for the Jacobian matrix,* $[\mathbb{J}_{jk}] = [\partial(A)_j/\partial x^k]$*, plays the role of the rescaled molar density* $\rho/\rho_c$ *in thermodynamics.*

The critical point in 3D occurs for the set $\{X_M = 3,\ Y_G = 3,\ Z_A = 1,\ T_K = 0\}$ with $\xi \Leftrightarrow \widetilde{\rho} = [1,1,1,0]$. A comparison of the universal equation and the van der Waals gas equation yields

$$3 = X_M \quad (2.50)$$

$$\{8\widetilde{T} + \widetilde{P}\}/3 = Y_G, \quad (2.51)$$

$$\widetilde{P} = Z_A, \quad (2.52)$$

For the classic van der Waals gas, it is apparent that the (linear) similarity invariant (which is composed of the sum of molar density eigenvalues) is at its critical point value, $X_M = 3$. The (quadratic) similarity invariant is equal to $Y_G = (\{8\widetilde{T}+\widetilde{P}\}/3)$ and is composed of both temperature and pressure terms. The Adjoint (cubic) interaction similarity invariant is equal to $Z_A = \widetilde{P}$, the rescaled Pressure. It is remarkable that for the van der Waals gas, the linear terms (representing string or surface tension effects, have been fixed at their "Critical Point" values. These concepts will be presented in more detail in chapter 3.

If a further reduction in dimension occurs to Pfaff dimension 2, (with 2 null eigenvalues) the Phase equation with $\{M,G\}$reduces to

$$\Theta(x,y;z,t\ \xi) = (\xi^2 - X_M\xi^1 + X_G)\xi^2 \Rightarrow 0, \quad (2.53)$$

$$\text{with } X_M = (\xi_1 + \xi_2), \quad (2.54)$$

$$Y_G = (\xi_1\xi_2), \quad (2.55)$$

The critical point in 2D occurs for the set $\{X_M = 2,\ Y_G = 1,\ Z_A = 0, T_K = 0\}$ with $\xi_k = [1,1,0,0]$.

### The reduced Phase function

There exists a well known transformation of complex variable which will reformulate the characteristic polynomial. Substitute $\xi = s + M/4$. The result is a new "reduced" Phase polynomial $\Phi(x,y,z,t;\ s) = \Theta(x,y,z,t;\ \xi)_{reduced}$ of the form

$$\Phi(x,y,z,t;\ s) = s^4 + gs^2 - as + k = 0. \quad (2.56)$$

$$g = (-3X_M^2/8 + Y_G) \quad (2.57)$$

$$a = (X_M/2)^3 - Y_G X_M/2 + Z_A) \quad (2.58)$$

$$k = T_K - Z_A(X_M/4) + Y_G(X_M/4)^2 - 3(X_M/4)^4 \quad (2.59)$$

$$s = \xi - X_M/4 \quad (2.60)$$

The "reduced" Phase function is not the same as the "rescaled" Phase function. The coefficients $\{g, a, k\}$ are constructed from the real numbers $\{X_M, Y_G, Z_A, T_K\}$, of the reduced Phase polynomial. Hence polynomial analysis implies that the eigenvalues of the reduced Phase function belong to 3 equivalence classes (See discussion following eq. 2.11).

For a van der Waals gas ($X_M = 3,\ T_K = 0$), the reduced coefficients become

$$\text{Van der Waals gas} \quad (2.61)$$

$$g = -27/8 + Y_G = -27/8 + \{8\widetilde{T} + \widetilde{P}\}/3 \quad (2.62)$$

$$a = -27/8 + \{8\widetilde{T} - \widetilde{P}\}/2 \quad (2.63)$$

$$k = -243/256 + T_K - 9/16\widetilde{P} + 3/2\widetilde{T}, \quad (2.64)$$

$$s = \xi - 3/4. \quad (2.65)$$

The critical point has been moved to $s = 1/4$ for the van der Waals gas, as one of the eigenvalues is presumed to be zero. The reduced formula has eliminated the cubic term in the universal phase function by displacing the critical point to the origin in terms of the variable $s$, if all eigenvalues are not zero.

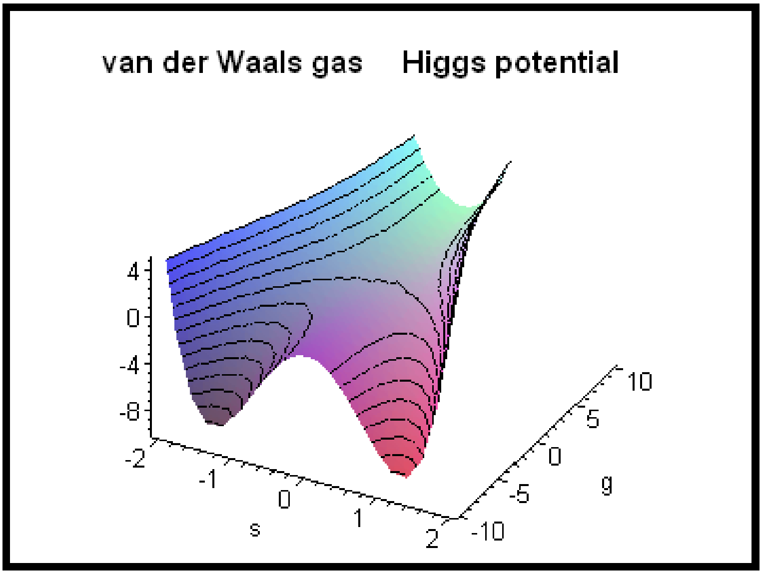
Consider the reduced Phase formula, and its derivatives with respect to the family parameter, $s$.

$$\Phi = s^4 + gs^2 - as + k = 0, \tag{2.66}$$
$$\therefore \quad k = -(s^4 + gs^2 - as), \tag{2.67}$$
$$\Phi_s = \partial\Phi/\partial s = 4s^3 + 2gs - a = 0 \tag{2.68}$$
$$\therefore \quad a = 4s^3 + 2gs \tag{2.69}$$
$$\Phi_{ss} = \Phi_s = \partial^2\Phi/\partial s^2 = 12s^2 + 2g = 0 \tag{2.70}$$
$$\therefore \quad g = -6s^2 \tag{2.71}$$

Replacing the parameter $a$ (from the envelope condition, $\Phi_s = 0$) in the equation for $k$ yields

$$\text{Thermodynamic Higgs Potential} \quad k = s^2(3s^2 + g). \tag{2.72}$$

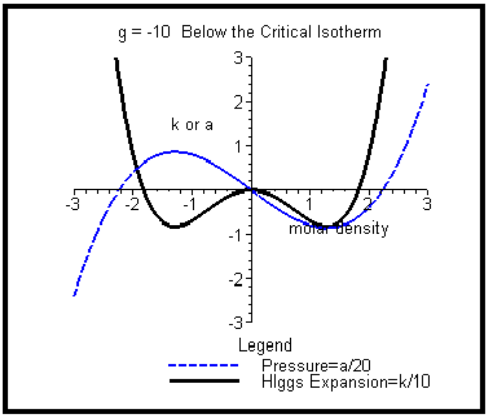
A plot of the equation for $k$ is given below for various g and s,. The 4th order shape of the function motivates the name "Thermodynamic Higgs Potential"

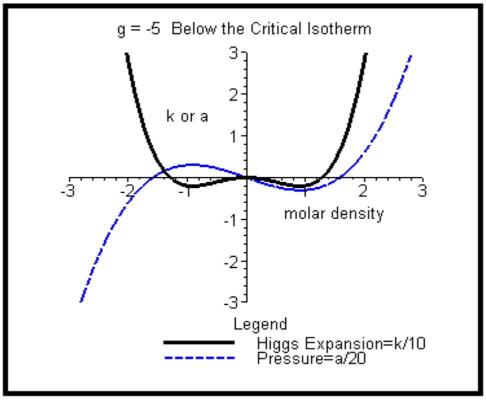


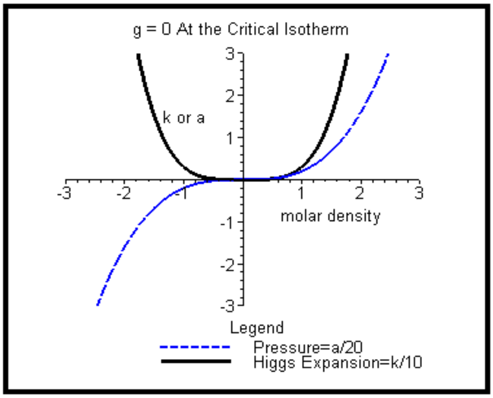

**The Higgs potential is a Universal Envelope.**

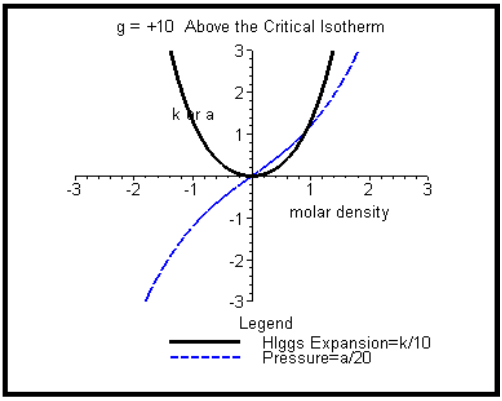
In the general case with $T_K \neq 0$, then g $= 0$, s=0, k=0, represents the "critical point". Note that the set k=0 defines a pitchfork bifurcation. For g (~reduced temperature) values below the critical point, the function $k$ is a polynomial of 4th degree, but above the "critical temperature" the function $k$ is quadratic. It is evident that below the critical isotherm, the "expansion" term $k$ can have both negative and positive values. The formula for the 4D expansion coefficient therefor can also have positive or negative values. The quartic "potential" is reminiscent of the "Higgs" potential in relativistic field theories and the "Landau" potential in mean field theories. Note that these properties have been obtained without explicit use of a metric or connection, nor quantum mechanics.

From the van der Waals theory, the first partial derivative of the classic phase function yields the Pressure. For the universal Phase polynomial the pressure is determined by the equation $\Phi_s = 0$, Indeed, the formula $a = 4s^3 + 2gs$ yields the universal equation for $a$ (the Pressure) in terms of the molar density "$s$". A plot of $a$ (Pressure) versus $s$ (molar density) at fixed $g$ (temperature) gives the familiar cubic shape, deformably equivalent to the van der Waals gas. For the critical temperature ($g = 0$) the shape of the critical isotherm is exactly the same as for the critical isotherm of the van der Waals gas. Both $k$ (Expansion in dashed blue) and $a$ (Pressure in solid black) are presented in the following diagrams as constant g (~temperature) slices above and below the critical point.




g = -5 Below the Critical Isotherm
k or a
molar density
Legend
Higgs Expansion=k/10
Pressure=a/20


g = 0 At the Critical Isotherm
k or a
molar density
Legend
Pressure=a/20
Higgs Expansion=k/10


g = +10 Above the Critical Isotherm
k or a
molar density
Legend
HIggs Expansion=k/10
Pressure=a/20

### The Critical Isotherm

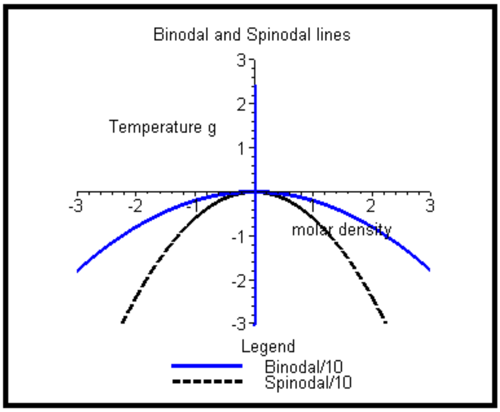
The topology of the quartic phase (potential) function is separated by a critical isotherm into two sectors. For temperatures below the critical temperature, the quartic formula yields a Higgs-like sector where expansion properties $k$ are negative, and where liquid and vapor phases can coexist. Above the critical temperature the 4th order expansion properties $k$ are positive, and the sector has lost its Higgs-like properties. The critical isotherm, $g = 0 = (-3X_M^2/8 + Y_G)$ defines a line of singularities separating the two sectors. The shape of the curve mimics the dividing center of hysteresis phenomena.

### The Binodal line

The zero sets of the "reduced Pressure $(a)$" occur only for temperatures below the critical point and are described by the solution formula, $0 = 4s^3 + 2gs$. Along with the solution $s = 0$ coming from the second factor in the general phase formula for zero k, $\Phi = s^4 + gs^2 - as = 0$, a plot of the zeros of the "reduced Pressure $(a)$" in the $s - g$ plane yield the Binodal line as a pitchfork bifurcation, with the transition occurring at the critical temperature. From the van der Waals gas model, the Binodal line delineates the single phase from the mixed phase regions. The Pitchfork is essentially the line of zero first partial derivatives of the Higgs sector of the universal phase function. This result appears to be the first non phenomenological derivation of the Binodal line. These Pitchfork features are readily seen in the previous figure giving a 3D version of the Higgs - van der Waals gas potential.

### The Spinodal Line

A second piece of topological information can be obtained from those points where the partial derivative of the pressure vanishes. These points are given by solutions to the equation $\Phi_{ss} = 12s^2 + 2g = 0$. Again only for temperatures $g$ below the critical point will the formula give a set of points that describes classically what has been called the Spinodal line. In van der Waals theory the Spinodal line defines the "limit" of single phase stability and can only be realized transiently, in the absence of fluctuations. Both Spinodal line (blue) and the Binodal line (black) are plotted in the next figure




The Binodal line and the Spinodal line can be related to homology invariants of projective transformations

### *2.3.4 Oscillations and the Hopf bifurcation*

When the eigen values of the characteristic polynomial are pure imaginary, Hopf oscillations can occur. Suppose that the complex eigenvalues are represented as $\{i\alpha, -i\alpha, i\beta, -i\beta\}$. Then the similarity invariants are $X_M = 0, Y_G = \alpha^2 + \beta^2, Z_A = 0, T_K = \alpha^2\beta^2$. Hence the criteria for a double Hopf oscillation frequency requires that the algebraically odd similarity invariants vanish and the algebraically even similarity invariants are positive definite. (Recall that in the 3D theory of minimal 2D surfaces, the mean curvature is related to the linear similarity invariant, $X_M \Rightarrow 0$). For a single dominant Hopf oscillation frequency ($\beta \Rightarrow 0$), the Hopf conditions are: $X_M = 0, Y_G \Rightarrow \alpha^2 > 0, Z_A = 0, T_K \Rightarrow 0$. These conditions can be computed relatively easily, and will be demonstrated in the examples below. Note that the minimal hypersurface condition $X_M \Rightarrow 0$ may be satisfied by states with $Y_G < 0$ in accord with the examples of soap films. Such conditions are related to non oscillatory solitons which form "stationary states", but are globally stabilized far from equilibrium. (See Chapter 1 on Falaco Solitons.)

### *2.3.5 Minimal surfaces*

The Universal Phase function, $\Theta$, may be considered as a family of implicit hypersurfaces in the 4 dimensional space, $\{X_M, Y_G, Z_A, T_K\}$ with a complex family (order) parameter, $\xi$. Moreover, it should be realized that the Universal Phase Function is a holomorphic function, $\Theta = \phi + i\chi$ in the complex variable $\xi = u + iv$. That is

$$\Theta(X_M, Y_G, Z_A, T_K;\ \xi) \Rightarrow \phi + i\chi, \tag{2.73}$$

where

$$\begin{aligned} \phi &= u^4 - 6u^2v^2 + v^4 - X_M(u^2 - 3v^2)u \\ &\quad + Y_G(u^2 - v^2) - Z_A u + T_K \qquad (2.74) \\ \chi &= 4(u^2 - v^2)uv - X_M(3u^2 - v^2)v + 2Y_G uv - Z_A v, \qquad (2.75) \end{aligned}$$

As such, in the 4D space of two complex variable pairs, $\{\phi + i\chi, u + iv\}$, according to the theorem of Sophus Lie, any such holomorphic function produces a pair of conjugate *minimal* surfaces in the 4 dimensional space $\{\phi, \chi, u, v\}$. It follows that there exist a sequence of maps,

$$\{x, y, z, t\} \Rightarrow \{X_M, Y_G, Z_A, T_K\} \Rightarrow \{\phi, \chi, u, v\} \tag{2.76}$$

such that the family of hypersurfaces can be decomposed into a pair of conjugate minimal surface components.

For a phase function generated by the constraints, $X_M = Z_A \Rightarrow 0$, the minimal surface functions become defined by the equations

$$\begin{aligned} \phi &= u^4 - 6u^2v^2 + v^4 + Y_G(u^2 - v^2) + T_K \qquad (2.77) \\ \chi &= 4(u^2 - v^2)uv + 2Y_G uv. \qquad (2.78) \end{aligned}$$

For the Hopf Map the eigenvalues are pure imaginary, hence

$$\begin{aligned} \phi_{Hopf} &= +v^4 + Y_G(-v^2) + T_K \qquad (2.79) \\ \chi &= 0. \qquad (2.80) \end{aligned}$$

It is important to realize that the similarity invariant $X_M \Rightarrow 0$ does not define a minimal surface unless the Jacobian matrix of the 1-form is scaled by the Gauss map.

### Examples of conjugate pairs of minimal surfaces

The idea is that the complex position vector, $\mathbf{V} = [U, V, W]$, whose real or imaginary parts will map out a minimal surface in 3D, can be generated from the Weierstrass representation [146] in terms of the holomorphic function $H(\varpi) = \phi + i\chi$,

$$X(\varpi) = \int(1-\varpi^2)H(\varpi)d\varpi \tag{2.81}$$
$$Y(\varpi) = \int(1+\varpi^2)H(\varpi)d\varpi \tag{2.82}$$
$$Z(\varpi) = \int(2\varpi)H(\varpi)d\varpi \tag{2.83}$$

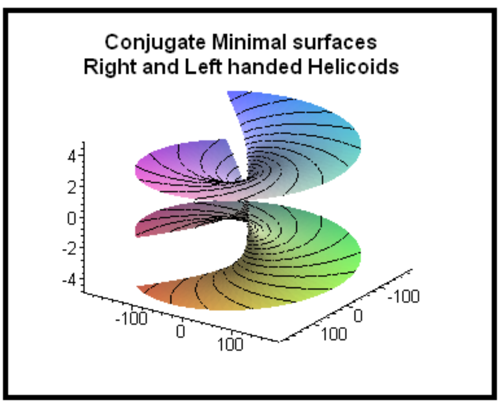
Rewriting $H(\varpi)$ in the form

$$H(\varpi) = (b-ia)/2\varpi^2, \quad with \quad \varpi = -i\ exp(\eta + i\xi) \tag{2.84}$$

and substituting into the Weierstrass formulas yields the position vector to a family of minimal surfaces of the form

$$X = a\ sinh(\eta)\ cos(\xi) - b\ cosh(\eta) sin(\xi) \tag{2.85}$$
$$Y = a\ sinh(\eta) sin(\xi) - b\ cosh(\eta) cos(\xi) \tag{2.86}$$
$$Z = a\,\eta + b\,\xi \tag{2.87}$$

For $a = 0$ the surface is a catenoid; for $b = 0$ the surface is a helicoid. (see p.70 in [144]). For $a$ and $b$ non zero, the minimal surface so generated consists of two conjugate minimal surfaces intertwined (the example has $a = b = .5$)




Note that the conjugate pairs have different chirality. Other examples of such conjugate pairs are displayed below.

**Example of a fractal minimal surface**

As a second example of the Sophus Lie theorem, consider the Holomorphic function and its functional iterates

$$H_1(\varpi) = \ (\varpi^2 - D), \ \ H_2(\varpi) = \ ((\varpi^2 - D)^2 - D), \ ... \tag{2.88}$$

According to the minimal surface theorem, this Holomorphic function represents a one (complex) parameter family of minimal surfaces in 4-dimensions. It follows that the Mandelbrot set , which is given by the values of $D$ for which the function $H_1(\varpi)$ fails to iterate the origin ($\varpi = 0$) to infinity is the fractal envelope of a family of minimal surfaces in 4-dimensions parameterized by $D = a + ib$. The compliment to the Mandelbrot set is a minimal surface with a fractal boundary where all functional sequences iterate to infinity. Hence the "fractal" minimal surface is complete. The non-intuitive conclusion is that a minimal surface can be fractal!

**The "Gibbs entropy" minimal surface**

As another surprising example, consider those functions of a complex variable such that $H(\varpi) = (\partial F(\varpi)/\partial\varpi)^3$. All functions $F(\varpi)$ that have the form

$$F(\varpi) = \{\alpha \ \varpi \ ln(\varpi) + C \ \varpi\} + (B - D \ \varpi^2) \tag{2.89}$$

$$= \{Gibbs \ Entropy\} + (Mandelbrot \ generator) \tag{2.90}$$

generate the same Weierstrass function,

$$H(\varpi) = (\partial^3 F(\varpi)/\partial\varpi^3) = 2\alpha/\varpi^2. \tag{2.91}$$

The format of $F(\varpi)$ is strikingly reminiscent of those formulas that appear in the literature to describe the Gibb's entropy. The coefficients $\alpha, B, C$ and $D$ are presumed to be complex constants. Rewriting $H(\varpi)$ in the form

$$H(\varpi) = (b - ia)/2\varpi^2, \ \ with \ \ \ \varpi = -i \ exp(\eta + i\xi) \tag{2.92}$$

and substituting into the Weierstrass formulas yields the position vector to a family of minimal surfaces. When $\alpha$ is real then extremal minimal surface is a catenoid again; when $\alpha$ is imaginary, the minimal surface is a helix. When $\alpha$ is complex, the result is a pair of conjugate helicoids.

The interesting features are :

1. All complex wave functions are related to a minimal surface by this technique.

2. The primitive function $F(\varpi)$ is related to the Helmholtz free energy, and it is the "entropy" term, $\alpha\ \varpi\ ln(\varpi)$ that generates the family of minimal surfaces known as the conjugate helicoids. (The topic of conjugate helicoids will be revisited below.)

3. The resulting minimal surface is independent of the linear term $C\ \varpi$ and the "Mandelbrot germ", $(B - D\ \varpi^2)$.

4. The Petrov type D classifications (which yield the only known black hole solutions to the Einstein gravity theory [42]) are related to minimal surfaces.

**Envelopes**

The theory of implicit hypersurfaces focuses attention upon the possibility that the Universal Phase function has an envelope. The existence of an envelope depends upon the possibility of finding a simultaneous solution to the two implicit surface equations of the family:

$$\Theta(x, y, z, t;\ \xi) = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi + T_K \Rightarrow 0, \quad (2.93)$$

$$\partial\Theta/\partial\xi = \Theta_\xi = 4\xi^3 - 3X_M\xi^2 + 2Y_G\xi - Z_A \Rightarrow 0, \quad (2.94)$$

For the envelope to be smooth, it must be true that $\partial^2\Theta/\partial\xi^2 = \Theta_{\xi\xi} \neq 0$, and that the exterior 2-form, $d\Theta\hat{\ }d\Theta_\xi \neq 0$ subject to the constraint that the family parameter is a constant: $d\xi = 0$. The envelope as a smooth hypersurface does not exist unless both conditions are satisfied (see Chapter 7).

The envelope is determined (to within a factor) by the discriminant of the Phase Function polynomial, which, as a zero set, is equal to a universal implicit hypersurface, $DISC\Theta \Rightarrow 0$, in the 4 dimensional space of similarity variables $\{X_M, Y_G, Z_A, T_K\}$. This function can be written in terms of the similarity "coordinates" (suppressing the subscripts) as :

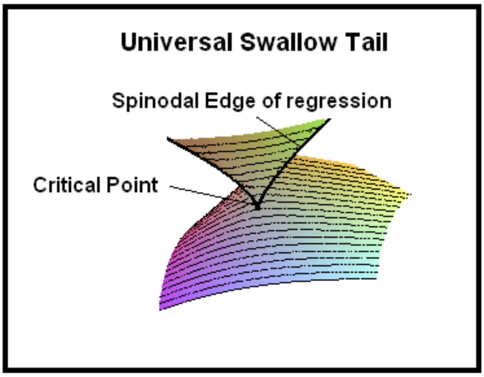
$$\begin{aligned} DISC\Theta = {} & 18X^3ZYT - 27Z^4 + Y^2X^2Z^2 - 4Y^3X^2T \\ & +144YX^2T^2 + 18XZ^3Y - 192XZT^2 \\ & -6X^2Z^2T + 144TZ^2Y - 4X^3Z^3 \\ & -27X^4T^2 - 4Y^3Z^2 + 16Y^4T \\ & -128Y^2T^2 + 256T^3 - 80XZY^2T. \end{aligned} \quad (2.95)$$

The discriminant (envelope) has eliminated the family order parameter, $\xi$.

An alternate formulation describes the discriminant of the Reduced Phase phase function, $\Phi = \Theta_{reduced}$:

$$DISC\Phi := -27a^4 + 4(-g^2 + 36k)ga^2 + 16k(4k - g^2)^2. \qquad (2.96)$$

The hypersurface defined by the discriminant of the phase function $\Theta_{reduced} = \Phi$ yields the (symmetrized) version of the universal swallow tail hypersurface. A plot of the universal envelope $\Phi = 0$ (in terms of the coordinates $(g,\ a,\ k)$ is given in the following Figure.




It is apparent that the van der Waals gas is a deformation of the universal swallowtail hypersurface formed as the envelope of the reduced phase function, $\Theta_{reduced}$. It is remarkable that the Discriminant Envelope of the the universal phase function, $\Theta$, and the Discriminant Envelope of the reduced phase function, $\Phi$, are the same (in the space of coordinates $\{X_M, Y_G, Z_A, T_K\}$).

Remarkably, choosing the constraint condition in terms of the hypothetical condition that the Mean similarity invariant Curvature vanishes, $X_M \Rightarrow 0$, leads to a domain in the 4D space where the reduced discriminant defines a universal swallow tail surface homeomorphic (deformable) to the Gibbs surface of a van der Waals gas (subscripts suppressed):

$$\begin{aligned}
\text{Minimal Surface} \quad &: \quad \textbf{Universal Swallowtail Envelope} \;\; X_M \Rightarrow 0 \\
DISC\Theta_{(X_M=0)} \quad &= \quad -27Z^4 + 144TZ^2Y - 4Y^3Z^2 + 16Y^4T \\
&\quad -128Y^2T^2 + 256T^3 \qquad (2.97) \\
&\approx \quad DISC\Theta_{reduced} = DISC\Phi \Rightarrow 0. \qquad (2.98)
\end{aligned}$$

It must be remembered that this Minimal surface is a hypersurface in the space of Pfaff topological dimension 4. Examples are given in that which follows. In other words, the Gibbs function for a van der Waals gas is a universal idea associated with minimal hypersurfaces, $X_M = 0$, of thermodynamic systems of Pfaff topological dimension 4. The similarity coordinate $T_K$ plays the role of the Gibbs free energy, in terms of the Pressure ($\tilde{}Z_A$) and the Temperature ($\tilde{}Y_G$). The Spinodal line as a limit of phase stability, and the critical point are ideas that come from the study of a van der Waals gas, but herein it is apparent that these concepts are universal topological concepts that remain invariant with respect to deformations

Another choice would be to constrain the envelope such that it resides in a domain where the 1-form of Action is of Pfaff topological dimension 3. The physical system is closed, but it is not necessarily in equilibrium. An equilibrium or isolated physical system consists of a single topological component, or phase (the Cartan topology is a connected topology). Domains where the Pfaff topological dimension represent mixed phases imply more than 1 topological component, and are to be associated with regions where the Pfaff topological dimension is $\geq 3$. The case of Pfaff dimension 3 would correspond to regions where the 3-form of Topological Torsion is not zero (the Cartan topology becomes a disconnected topology - See Chapter 4). Such non equilibrium domains correspond to the situation where the determinant of the $4 \times 4$ Jacobian matrix vanishes. That is, set $T_K = 0$, to obtain the (3D constrained) envelope $DISC_{(T_K=0)}$:

$$DISC_{(T_K=0)} = Z^2\{-4X^3Z + 18XZY + Y^2X^2 - 4Y^3 - 27Z^2\} \Rightarrow 0, \quad (2.99)$$

It is remarkable that the bracketed formula (in $X, Y, Z$ coordinates) is precisely the Cardano cubic formula that separates the topological features of the generalized cubic equation. It is important to recognize that the development of a universal non equilibrium van der Waals gas has not utilized the concepts of metric, connection, statistics, relativity, gauge symmetries, or quantum mechanics.

### The Edge of Regression and Self Intersections

The envelope is smooth as long as $\partial^2\Theta/\partial\Psi^2 = \Theta_{\xi\xi} \neq 0$, and that the exterior 2-form, $d\Theta\hat{}d\Theta_\xi \neq 0$ subject to the constraint that the family parameter is a constant: $d\xi = 0$. If $d\Theta\hat{}d\Theta_\xi \neq 0$, but $\Theta_{\xi\xi} = 0$, then the envelope has a self intersection singularity. If $d\Theta\hat{}d\Theta_\xi = 0$, but $\Theta_{\xi\xi} \neq 0$, there is no self intersection, and no envelope.

If the envelope exists, further singularities are determined by the higher order partial derivatives of the Universal Phase function with respect to $\xi$.

$$\begin{aligned} \partial^2\Theta/\partial\xi^2 &= \Theta_{\xi\xi} = 12\xi^2 - 6X_M\xi + 2Y_G, & (2.100) \\ \partial^3\Theta/\partial\xi^3 &= \Theta_{\xi\xi\xi} = 24\xi - 6X_M. & (2.101) \end{aligned}$$

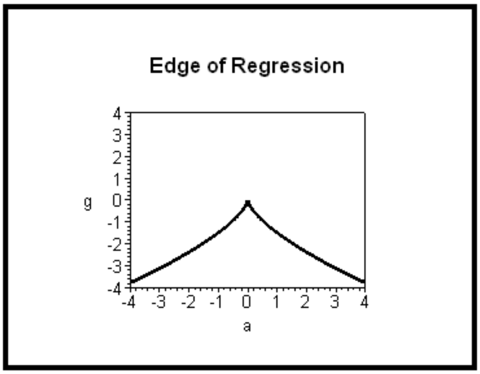
When $\partial^3\Theta/\partial\xi^3 = \Theta_{\xi\xi\xi} \neq 0$, and $d\Theta\hat{}d\Theta_\xi\hat{}d\Theta_{\xi\xi} \neq 0$, the envelope terminates in a edge of regression. The edge of regression is determined by the simultaneous solution of $\Theta = 0, \Theta_\xi = 0$ and $\Theta_{\xi\xi} = 0$. Solving for $\xi$ in $\Theta_{\xi\xi} = 0$ yields $Y_G = \xi(3X_M - \xi)$.

**Reduced Phase Functions**

Reconsider the reduced phase function, $\Phi$, in terms of coordinate coefficients $\{g, a, k\}$, and its partial derivatives with respect to the family parameter, $s$ :

$$\begin{aligned} \Phi &= s^4 + gs^2 - as + k = 0, & (2.102) \\ \Phi_s &= \partial\Phi/\partial s = 4s^3 + 2gs - a, & (2.103) \\ \Phi_{ss} &= \partial^2\Phi/\partial s^2 = 12s^2 + 2g, & (2.104) \\ DISC\Phi &= -27a^4 + 4(-g^2 + 36k)ga^2 & \\ &\quad +16k(4k - g^2)^2 & (2.105) \end{aligned}$$

The reduced formula is more tractable for, if the family parameter is fixed, then the equation represents a implicit surface in the space of coordinates, $\{g, a, k\}$. A representation for this implicit surface $DISC\Phi = 0$ was given in the previous figure. It is an obvious deformation equivalent to the Gibbs function for a van der Waals gas. The edge of regression is given by the zero set of $\Phi_{ss} = 0$ or $g = -6s^2$. Using this value in $\Phi_s = 0$ permits a solution for $a$ in terms of $s$. Using these values for $a$ and $g$ in $\Phi = 0$ gives the three components of a position vector $\mathbf{R}$ =$[-6s^2, -8s^3, -3s^4]$ in $\{g, a, k\}$ space for the edge of regression. The result for the edge of regression in the $g - a$ plane is plotted below:




The same function is plotted as the edge of regression for the Universal Swallow Tail in the previous Figure.

### Universal Phase Function Minimal Surfaces

For the minimal surface representation of the Gibbs surface for a van der Waals gas, the edge of regression defines the Spinodal line of ultimate phase stability. The edge of regression is evident in the Swallowtail figure (Figure 2.1) describing the Gibbs function for a van der Waals gas. If $\Theta_{\xi\xi} = 0$, then for $X_M = 0$ the envelope has a self intersection. It follows from $\Theta_{\xi\xi} = 0$, that $\xi^2 = -Y_G/6$, which when substituted into

$$\Theta_{\xi} = 4\xi^3 + 2Y_G\xi - Z_A \Rightarrow 0, \tag{2.106}$$

yields the

**Universal ($\mathbf{X_M}$=0) Gibbs Edge of Regression :** $Z_A^2 + Y_G^3(8/27) = 0,$ (2.107)

which defines the Spinodal line, of the *minimal* surface representation for a universal non equilibrium van der Waals gas, in terms of "similarity" coordinates.

Within the swallow tail region the "Gibbs" surface has 3 real roots; outside the swallow tail region there is a unique real root. The edge of regression furnished by the Cardano function defines the transition between real and imaginary root structures. The details of the universal non equilibrium van der Waals gas in terms of envelopes and edges of regression with complex molal densities or order parameters will be presented elsewhere. These systems are not equilibrium systems for the Pfaff dimension is not 2. Of obvious importance is the idea that the a zero value for both $Z_G$ and $T_K$ are required to reduce the Pfaff dimension to 2, the necessary condition for an equilibrium system.

### Ginsburg Landau Currents

With a change of notation ($\xi \Rightarrow \Psi$), the Universal Phase function can be solved for the determinant of the Jacobian matrix, which is equal to the similarity invariant $T_K$,

$$T_K = -\{\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi\}. \tag{2.108}$$

The similarity invariant $T_K$ represents the determinant of the Jacobian matrix. All determinants are in effect N - forms on the domain of independent variables. All N-forms can be related to the exterior derivative of some N-1 form or current, $J$. Hence

$$dJ = T_K\Omega_4 = (div\mathbf{J}+\partial\rho/\partial t)\Omega_4 = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi)\Omega_4. \tag{2.109}$$

For currents of the form

$$\mathbf{J} = grad\ \Psi, \tag{2.110}$$

$$\rho = \Psi, \tag{2.111}$$

the Universal Phase function generates the universal Ginsburg Landau equations

$$\nabla^2\Psi + \partial\Psi/\partial t = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi). \tag{2.112}$$

#### *2.3.6 Singularities as defects of Pfaff dimension 3*

The family of hypersurfaces can be topologically constrained such that the topological dimension is reduced, and/or constraints can be imposed upon functions of the similarity variables forcing them to vanish. Such regions in the 4 dimensional topological domain indicate topological defects or thermodynamic changes of phase. It is remarkable that for a given 1-form of Action there are an infinite number rescaling functions, $\lambda$, such that the Jacobian matrix $\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k]$ is singular (has a zero determinant). For if the coefficients of any 1-form of Action are rescaled by a divisor generated by the Holder norm,

$$\textbf{Holder Norm}: \quad \lambda = \{a(A_1)^p + b(A_2)^p + c(A_3)^p + e(A_4)^p\}^{m/p}, \tag{2.113}$$

then the rescaled Jacobian matrix

$$\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k] \tag{2.114}$$

will have a zero determinant, for any index p, any set of isotropy or signature constants, a, b, c, e, if the homogeneity index is equal to unity: $m = 1$. This homogeneous constraint implies that the similarity invariants become projective invariants, not just equi-affine invariants. Such species of topological defects can have the image of a 3-dimensional implicit characteristic hypersurface in space-time:

$$\textbf{Singular hypersurface in 4D}: \ \det[\partial(A/\lambda)_j/\partial x^k] \Rightarrow 0 \tag{2.115}$$

The singular fourth order Cayley-Hamilton polynomial of $[\mathbb{J}_{jk}]$ then will have a cubic polynomial factor with one zero eigenvalue.

For example, consider the simple case where the determinant of the Jacobian vanishes: $T_K \Rightarrow 0$. Then the Phase function becomes (for Pfaff Dimension 3):

$$\textbf{Universal Equation of State} \tag{2.116}$$

$$\Theta(\{X_M, Y_G, Z_A, T_K = 0\};\ \xi) \tag{2.117}$$

$$= \quad \xi(\xi^3 - X_M\xi^2 + Y_G\xi - Z_A) \Rightarrow 0. \tag{2.118}$$

The space has been topologically reduced to 3 dimensions (one eigen value is zero), and the zero set of the resulting singular Universal Phase function becomes a universal cubic equation that is homeomorphic to the cubic equation of state for a van der Waals gas.

When the rescaling factor $\lambda$ is chosen such that $p = 2, a = b = c = 1, m = 1$, then the Jacobian matrix, $[\mathbb{J}_{jk}]$, is equivalent to the "Shape" matrix for an implicit hypersurface in the theory of differential geometry. (See Chapter 8.) Recall that the homogeneous similarity invariants can be put into correspondence with the linear Mean curvature, $X_M \Rightarrow C_M$, the quadratic Gauss curvature, $Y_G \Rightarrow C_G$, and the cubic Adjoint curvature, $Z_A \Rightarrow C_A$, of the hypersurface. The characteristic cubic polynomial can be put into correspondence with a nonlinear extension of an ideal gas *not necessarily* in an equilibrium state.

### *2.3.7 The Adjoint Current and Topological Spin*

From the singular Jacobian matrix, $\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k]$, it is always possible to construct the Adjoint matrix as the matrix of cofactors transposed:

$$\text{Adjoint Matrix}: \ \left[\widehat{\mathbb{J}}^{kj}\right] = adjoint\left[\mathbb{J}_{jk}^{scaled}\right] \tag{2.119}$$

When this matrix is multiplied times the rescaled covector components, the result is the production of an adjoint current,

$$\text{Adjoint current}: \left|\widehat{\mathbf{J}}^k\right\rangle = \left[\widehat{\mathbb{J}}^{kj}\right] \circ |\mathbf{A}_j/\lambda\rangle \tag{2.120}$$

It is remarkable that the construction is such that the Adjoint current 3-form, if not zero, has zero divergence globally:

$$\widehat{J} = i(\widehat{\mathbf{J}}^k)\Omega_4 \tag{2.121}$$

$$d\widehat{J} = 0. \tag{2.122}$$

From the realization that the Adjoint matrix may admit a non zero globally conserved 3-form density, or current, $\widehat{J}$, it follows abstractly that there exists a 2-form density of "excitations", $\widehat{G}$, such that

$$\textbf{Adjoint current}: \widehat{J} = d\widehat{G}. \tag{2.123}$$

$\widehat{G}$ is not uniquely defined in terms of the adjoint current, for $\widehat{G}$ could have closed components (gauge additions $\widehat{G}_c$, such that $d\widehat{G}_c = 0$), which do not contribute to the current, $\widehat{J}$.

From the topological theory of electromagnetism [207] [200] there exists a fundamental 3-form, $A\hat{}G$, defined as the "topological Spin" 3-form,

$$\textbf{Topological Spin 3-form} : A\hat{}G. \tag{2.124}$$

The exterior derivative of this 3-form produces a 4-form, with a coefficient energy density function that is composed of two parts:

$$d(A\hat{}G) = F\hat{}G - A\hat{}\widehat{J}. \tag{2.125}$$

The first term is twice the difference between the "magnetic" and the "electric" energy density, and is a factor of 2 times the Lagrangian usually chosen for the electromagnetic field in classic field theory:

$$\textbf{Lagrangian Field energy density}: F\hat{}G = \mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E} \tag{2.126}$$

The second term is defined as the "interaction energy density"

$$\textbf{Interaction energy density}: A\hat{}\widehat{J} = \mathbf{A} \circ \widehat{\mathbf{J}} - \rho\phi. \tag{2.127}$$

For the special (Gauss) choice of integrating denominator, $\lambda$ with $(p = 2, a = b = c = 1, m = 1)$ it can be demonstrated that the Jacobian similarity invariants are equal to the classic Mean, Gauss, and Adjoint curvatures:

$$\{X_M, Y_G, Z_A, T_K\} \Rightarrow \{4M_{(linear)}, 6G_{(quadratic)}, 4A^{\star}_{(cubic)}, 0\}. \tag{2.128}$$

It can be demonstrated (with the use of Maple) that the interaction density is exactly equal to the Adjoint curvature energy density [204]:

$$\textbf{Interaction energy } A\hat{}\widehat{J} = 4A^{\star}_{(cubic)}\Omega_4 \quad \text{(Adjoint Cubic Curvature).} \tag{2.129}$$

The conclusion reached is that a non zero interaction energy density implies the thermodynamic system is not in an equilibrium state (but it could be in a "steady state" far from equilibrium).

However, it is always possible to construct the 3-form, $\widehat{S}$ :

$$\textbf{Topological Spin 3-form} : \widehat{S} = A\hat{}\widehat{G} \tag{2.130}$$

The exterior derivative of this 3-form leads to a cohomological structural equation similar the first law of thermodynamics, but useful for non equilibrium systems. This result, now recognized as a statement applicable to non equilibrium thermodynamic processes, was defined as the "Intrinsic Transport Theorem" in 1969 [164] :

$$\begin{aligned} &\textbf{Intrinsic Transport Theorem} && : \\ &\textbf{(Spin)}\ d\widehat{S} && = F\hat{}\widehat{G} - A\hat{}\widehat{J}, && (2.131) \\ &\textbf{First Law of Thermodynamics} && : \\ &\textbf{(Energy)}\ dU && = Q - W && (2.132) \end{aligned}$$

If one considers a collapsing system, then the geometric curvatures increase with smaller scales. If Gauss quadratic curvature, $6G_{(gauss\_quadratic)}$, is to be related to gravitational collapse of matter, then at some level of smaller scales a term cubic in curvatures, $4A^{\star}_{(adjoint\_cubic)}$, would dominate. It is conjectured that the cubic curvature produced by the interaction energy effect described above could prevent the collapse to a black hole. Cosmologists and relativists apparently have ignored such cubic curvature effects associated with non equilibrium thermodynamic systems.

### *2.3.8 Non Equilibrium Examples.*

In order to demonstrate content to the thermodynamic topological theory, two algebraically simple examples are presented below. (The algebra can become tedious for the rescaled Action 1-forms. Maple programs can be found in Vol. 6, "Maple programs for non Equilibrium systems". See HopfPhase.mws and Holder4d.mws.) The first corresponds to a Jacobian characteristic equation that has a cubic polynomial factor, and hence can be identified with a van der Waals gas. The second example exhibits the features associated with a Hopf bifurcation, where the characteristic equation has a quadratic factor with two pure imaginary roots, and two null roots.

**Example 1: van der Waals properties from rotation and contraction**

In this example, the Action 1-form is presumed to be of the form

$$A_0 = a(ydx - xdy) + b(tdz + zdt). \tag{2.133}$$

The 1-form of Potentials depends on the coefficients $a$ and $b$. The similarity invariants of the Jacobian matrix , $\mathbb{J}\,[(A_0)]$, formed from $A_0$, are:

Based on the 1-form $A_0$

$$\begin{aligned} X_M &= 0, && (2.134)\\ Y_G &= a^2 - b^2 && (2.135)\\ Z_A &= 0 && (2.136)\\ T_K &= -a^2 b^2 && (2.137)\end{aligned}$$

The eigen values of the Jacobian matrix are global complex constants: $\pm b$, $\pm\sqrt{-1}a$. If the 1-form of Action is rescaled by the Gauss map

$$\begin{aligned} A_0 &\Rightarrow A = A_0/\sqrt{(ax)^2 + (ay)^2 + (bz)^2 + (bt)^2} && (2.138)\\ r^2 &= (ax)^2 + (ay)^2 + (bz)^2 + (bt)^2 && (2.139)\end{aligned}$$

then the Jacobian matrix becomes the equivalent of the shape matrix, and the similarity invariants of the shape matrix are related to the *average* curvatures of the implicit Phase hypersurface, in a space of 1 less dimension. In the 3D subspace induced by the Gauss map ($\xi_4 = 0$) the shape matrix gives:

$$\begin{aligned} \text{Linear Mean curvature} &: C_M = X_M/3 && (2.140)\\ &= (\xi_1 + \xi_2 + \xi_3)/3 && (2.141)\\ \text{Quadratic Gauss curvature} &: C_G = Y_G/3 && (2.142)\\ &= (\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1)/3 && (2.143)\\ \text{Cubic Adjoint curvature} &: C_A = Z_A && (2.144)\\ &= \xi_1\xi_2\xi_3 && (2.145)\\ \text{Quartic Curvature :} &\quad C_K = 0. && (2.146)\end{aligned}$$

The computations for the given 1-form of Action yield the results:

$$\begin{aligned}
\text{Based on the 1-form } A &: \quad \text{Gauss map scaling} \\
\text{Linear Mean curvature} &: \quad C_M = -2b^3tz/(r^2)^{3/2} && (2.147) \\
\text{Quadratic Gauss curvature} &: \quad C_G = -a^2b^2\{(x^2+y^2) && (2.148) \\
& \quad -(z^2+t^2)\}/(r^2)^2 \\
\text{Cubic Adjoint curvature} &: \quad C_A = -2a^2b^3tz/(r^2)^{5/2} && (2.149) \\
\text{Quartic Curvature :} & \quad C_K = 0. && (2.150)
\end{aligned}$$

The Determinant (4th order curvature) vanishes by construction of the renormalization in terms of the Gauss map. This null result does not mean the Pfaff dimension of $A$ is less than 4 globally, but the constraint defines a singular set upon which there is a closed Current. This current is the Adjoint current of the previous section.

However, the rescaled 1-form $A$ is still of Pfaff dimension 4 and has a non zero topological torsion 3-form and a non zero topological torsion 4 form:

$$Top\_Torsion = 2ab \cdot [0,0,-z,t]/(r^2) \qquad (2.151)$$

$$\text{Pfaff Dimension 4} : dA\hat{\ }dA = 4b^3a(t^2-z^2)/(r^2)^2\ \Omega_4 \qquad (2.152)$$

The Gauss map permits the construction of the "Adjoint conserved current", which combined with the components of the Action 1-form yield an interaction energy density exactly equal to the cubic curvature $C_A$.

$$\text{Adjoint Current} : \mathbf{J}_s = ([x,y,z,t])\ /(r^2)^2, \qquad (2.153)$$

$$\text{interaction energy density:} \quad \mathbf{A} \circ \mathbf{J}_s - \rho\phi = C_A. \qquad (2.154)$$

The rescaled Jacobian matrix has 1 zero eigen value and 3 non zero eigenvalues. Hence, the cubic polynomial will yield an interpretation as a van der Waals gas. The Adjoint current represents a contraction in space-time, while the flow associated with the 1-form has a rotational component about the z axis.

**Example 2: A Hopf 1-form**

In this example,the Hopf 1-form is presumed to be of the form

$$A_0 = a(ydx - xdy) + b(tdz - zdt). \qquad (2.155)$$

The 1-form of Potentials depends on the chirality coefficients $a$ and $b$. There are two cases corresponding to left and right handed "polarizations": $a = b$ or $a = -b$. The results of the topological theory are :

$$\begin{aligned}
\text{Based on the 1-form} &: A_0 \\
X_M &= 0, && (2.156)\\
Y_G &= a^2 + b^2 && (2.157)\\
Z_A &= 0 && (2.158)\\
T_K &= a^2 b^2 && (2.159)\\
\text{Eigenvalues} &: \pm\sqrt{-1}b, \pm\sqrt{-1}a && (2.160)\\
\text{Torsion Current} &= [x, y, z, t]ab, && (2.161)\\
\text{Parity} &= 4ab && (2.162)
\end{aligned}$$

The 4 eigenvalues come in two imaginary pairs. The elements of each pair are equal and opposite in sign.

What is remarkable for this Action 1-form is that both the linear similarity invariant $X_M$ and the cubic similarity invariant $Z_A$ of the implicit phase hypersurface in 4D vanish, for any real values of a or b. The quadratic similarity invariant is non zero, positive real and is equal to $a^2 = b^2$. The quartic similarity invariant $T_K$ is non zero, positive real and is equal to $a^2 b^2$. The 1-form also supports a Topological Torsion current, with a non zero divergence.

However, if the 1-form $A_0$ is scaled by the Gauss map, the resulting Hopf implicit surface is a single 4D imaginary *minimal* two dimensional hyper surface in 4D and has two non zero imaginary curvatures, but a positive Gauss curvature! This a most unusual result, for the usual 2D minimal surface has equal and opposite real curvatures, with a negative Gauss curvature.

$$\begin{aligned}
\text{Based on the 1-form } A &: \text{Gauss map scaling} \\
\lambda^2 &= (ax)^2 + (ay)^2 + (bz)^2 + (bt)^2 && (2.163)\\
r &= \sqrt{x^2 + y^2 + z^2 + t^2)} && (2.164)\\
\text{Linear Mean curvature} &: C_M = 0 && (2.165)\\
\text{Quadratic Gauss curvature} &: C_G = +a^2 b^2\{r^2\}/(\lambda^2)^2 && (2.166)\\
\text{Cubic Adjoint curvature} &: C_A = 0 && (2.167)\\
\text{Quartic Curvature} &: C_K = 0 && (2.168)\\
\text{Eigenvalues} &: [0, 0, +\sqrt{-1}, -\sqrt{-1}](abr/\lambda^2). && (2.169)
\end{aligned}$$

Strangely enough the charge-current density induced by the Adjoint current is not zero, but it is proportional to the Topological Torsion vector that generates the 3 form $A\hat{}F$. The topological Parity 4 form is not zero, and depends on the sign of the coefficients a and b. In other words the 'handedness' of the different 1-forms determines the orientation of the normal field with respect to the implicit surface. It is known that a process described by a vector proportional to the topological torsion vector in a domain where the topological parity is non zero $4ba/(x^2+y^2+z^2+t^2) \neq 0$ is thermodynamically irreversible.

## 2.4 The Cosmological van der Waals Gas

The concepts of a universal phase function generated from a 1-form of Action $A$ for a non-equilibrium system (Pfaff Topological dimension $> 2$) will be applied to the construction of a cosmological model. It will be demonstrated how such a universal non-equilibrium van der Waals gas offers alternate explanations for the properties of our cosmological universe. In particular, the current "unexplained" concepts of dark energy and dark matter have a more classical foundation than is currently appreciated. Negative pressures are an enigma to many physicists, but are features well recognized by engineers who understand steam engines. Current relativity theories of gravity are based upon symmetric features of space time and in a sense "overlook" the anti-symmetric features of non-equilibrium thermodynamic systems.

From 1974 to the present, it has been the preoccupation of the present author to investigate the physical applications of irreversible topological evolution [165], [164], [173], [188]. This topic goes beyond the diffeomorphic equivalences of tensors, which can represent linearly connected processes that preserve the topology of the initial state during a transition to the final state, but which *cannot* be used to describe, deterministically, the thermodynamic irreversible processes of every day macroscopic reality. It became evident (due to inherent linearity restrictions) that the tensor analysis was inadequate to study irreversible topological evolution [172]. However, it was also noted that certain progress could be made by using methods inherent in Cartan's theory of exterior differential systems. In the theory of electromagnetism, it was known that ubiquitous tensor tools of metric and affine connection are useful, but not necessary, concepts [245]. Electromagnetism is indeed a topological theory, and has a universal expression in terms of two topological constraints on a set of exterior differential forms.

$$\text{Maxwell-Faraday:} \qquad F - dA = 0, \tag{2.170}$$

$$\text{Maxwell-Ampere:} \qquad J - dG = 0. \tag{2.171}$$

The resulting PDE's are covariant in form for any coordinate frame and in any number of dimensions greater than 3 [207].

The theory of thermodynamics is also a topological theory [235], independent from scales and deformations. Indeed the first law of thermodynamics is best understood as a topological constraint of cohomology, similar to the topological constraints that can be used to formulate Electromagnetism. The first law is a statement that the non-exact 1-form of heat, $Q$, minus the non-exact 1-form of work, $W$, is a perfect differential, $dU$ :

$$\text{First Law: } Q - W = dU. \tag{2.172}$$

To explain irreversible evolutionary processes, Lagrangian extremal methods are to be replaced by Cartan's Magic formula of continuous topological evolution acting upon physical systems that admit description in terms of exterior differential forms [190]. Exterior differential forms can carry global, topological information, and their use has led to definite progress in the understanding of thermodynamic irreversible turbulent flow, including the evolutionary creation of topological defects, or coherent structures, in irreversible dissipative hydrodynamic processes. These macroscopic continuous "condensation" concepts have both a micro and a cosmological realization. One of the most vivid experimental examples of such topological structures is given by the creation of Falaco solitons in a fluid surface of (density) discontinuity [178]. (Also see Chapter 1.) The visual evolutionary appearance of the swimming pool experiments leads to a suggestion that the creation of almost flat spiral arm galaxies from a non-linear dissipative cosmological fluid is also a feature of continuous topological evolution.

Topological evolution can take place by both continuous (cutting) and discontinuous (pasting) processes. The improper linear transformations (determinant = -1) of tensor analysis (such as mirror reflections) are not continuous about the identity. However, if the concept of tensor linear uniqueness is replaced by multivalued (but continuous about the identity) spinor transformations, the linear discontinuous but unique concepts admit equivalent descriptions in terms of continuous but non-unique topological evolution. A fundamental theme utilized herein is to replace the idea of discontinuous but unique with the concept of non-uniqueness but continuous.

The spirit of the idea is similar to the extension from the real line to the complex plane, where a (zero) point obstacle on the real line (yielding a discontinuity between positive or negative decrements) can be circumvented by a continuous (but multi-valued) right handed or left handed circuit about the (zero) obstacle in the complex plane. Note that a hole can be produced in a deformable disc by discontinuously cutting a hole and separating the parts, or by deforming the disc into the shape of the letter C and then (continuously) pasting the ends together. Two or more holes can be formed by discontinuously cutting a second hole, or by squeezing one hole to form the outline of a figure 8, and continuously pasting together the central region.

Remarkably, the fact the exterior differential forms could be homogeneous and evolve in a self similar manner permitted fractal structures to be admitted to the possible process descriptions of continuous topological evolution. The fact that the exterior differential systems may not be uniquely integrable (hence not in equilibrium) and yet could evolve into long lived states far from equilibrium became a mathematical fact, not just a philosophical dream. Moreover, it became possible to distinguish be chaos ( which can be thermodynamically reversible) and turbulence ( which is thermodynamically irreversible). Indeed, it became evident that thermodynamic irreversibility was an artifact of topological dimension 4 [199].

**Irreversible Processes and Topological Bulk Viscosity.**

When the Action for a physical system is of Pfaff dimension 4, there exists a unique direction field, $\mathbf{T}_4$, defined as the Topological Torsion 4-vector, that can be evaluated *entirely* in terms of those component functions of the 1-form of Action which define the physical system. To within a factor, this direction field[‡] has the four components of the 3-form $A\hat{}dA$, with the

[‡]A direction field is defined by the components of a vector field which establish the "line of action" of the vector in a projective sense. An arbitrary factor times the direction field defines the same projective line of action, just reparameterized. In metric based situations, the arbitrary factor can be interpreted as a renormalization factor.

following properties:

**Properties of the Topological Torsion vector** $\mathrm{T}_4$

$$
\begin{aligned}
i(\mathbf{T}_4)\Omega_4 &= A\hat{}dA && (2.173)\\
W &= i(\mathbf{T}_4)dA = \sigma\ A, && (2.174)\\
U &= i(\mathbf{T}_4)A = 0, && (2.175)\\
L_{(\mathbf{T}_4)}A &= \sigma\ A, && (2.176)\\
Q\hat{}dQ &= L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = \sigma^2 A\hat{}dA \neq 0 && (2.177)\\
dA\hat{}dA &= (2!)\ \sigma\ \Omega_4. && (2.178)
\end{aligned}
$$

Note that a $\mathbf{T}_4$ process is locally adiabatic, but not reversible.

Hence, by equation (4.125 ) evolution in the direction of $\mathbf{T}_4$ is thermodynamically irreversible, when $\sigma \neq 0$ and $A$ is of Pfaff topological dimension 4. The kernel of this vector field is defined as the zero set under the mapping induced by exterior differentiation. In engineering language, the kernel of this vector field are those point sets upon which the divergence of the vector field vanishes. The Pfaff topological dimension of the Action 1-form is 3 in the defect regions defined by the kernel of $\mathbf{T}_4$. The coefficient $\sigma$ can be interpreted as a measure of space-time volumetric expansion or contraction. It follows that both expansion and contraction processes (of space-time) are related to irreversible processes. It is here that contact is made with the phenomenological concept of "bulk" viscosity $= (2!)\sigma$. (For symplectic systems of higher Pfaff dimension $m = 2n + 2 \geq 4$, the numeric factor becomes $(m/2)!$.) It is important to note that the concept of an irreversible process depends on the square of the coefficient, $\sigma$. It follows that both expansion and contraction processes (of space-time) are related to irreversible processes. It is tempting to identify $\sigma^2$ with the concept of entropy production.

### Topological Evolution to Minimal Surfaces, Wakes and Spinors

During the period 1982 - 1995, [195], [197], [194], it also became apparent that long lived fluid dynamic wakes were related to minimal surfaces of tangential discontinuities. The argument was based on the fact that the dissipative Navier-Stokes equations could lead to long lived solutions, where the non-harmonic components of an initial velocity field would decay through viscous dissipation, leaving as residues, the harmonic components of the velocity field. This dissipative decay to long lived states far from equilibrium turns out to be a generic process of thermodynamic irreversibility [192].

The dissipative terms in the Navier - Stokes equations (neglecting compressibility) were proportional to the product of a viscosity coefficient times the vector Laplacian of the velocity field. As the harmonic components of the velocity field were precisely those components such that the vector Laplacian vanished, then no matter what the viscosity, the dissipation or decay of the harmonic components would be (almost) zero. Hydrodynamic wakes are essentially topological limit sets. Experience with differential geometry brought to mind the notion that the generator of a minimal surface was a harmonic vector field. Therefore wakes and minimal surfaces must be related concepts.

A more recent re-reading (in the spring of 2000) of Cartan's book on Spinors [39] (and Chandrasekhar's book on Black Holes [42]) lead to the thought that minimal surfaces and spinors were also related ideas – via the concept of an isotropic complex null vector. In fact, there is a connection between all of the ideas in the above abstract to this article. It is remarkable to me that both Cartan and Chandrasekhar do not mention the fact that an isotropic (complex null) vector is related to the generator of a Minimal Surface [146]. This is surprising to me, as Cartan was a differential geometer who knew about minimal surfaces. Cartan defined the "Spinor" as a *mapping* of a complex pair, $\{\alpha, \beta\}$ to a special 3 component complex vector, $\Sigma = [\sigma 1, \sigma 2, \sigma 3]$, in such a way that its quadratic form (sum of squares of the three components) is zero: $(\sigma 1)^2 + (\sigma 2)^2 + (\sigma 3)^2 = 0.$. This relationship of Spinor maps to minimal surfaces is ignored by many other authors, as well as Cartan and Chandrasekhar. A recent personal communication with Rindler (2000) also indicates that he also was not aware of the connection of these ideas. Evidently the idea of connecting Spinors and minimal surfaces was noticed by Dennis Sullivan (the topologist) about 1989. This reference I found (after I had stumbled on the idea independently) by an internet search which yielded the more recent article by R. Kusner and N. Schmitt [109]. There seems to be another publication relating spinors and minimal surfaces due to Dabrowski, a student of Bunovich, but the exact publication date (1986?) is not clear [57].

It would appear that many physicists and most engineers are not aware of the connection between spinors and minimal surfaces, and also their relationship to wakes and tangential discontinuities. The concept of a minimal surface yields an interesting and useful physical interpretation of spinors, especially as the interpretation does not depend explicitly upon quantum mechanical ideas, nor relativistic ideas, nor concepts of scale. The bottom line is that spinors have application to the engineering sciences at all scales, as well as to the microscopic world of Fermions and Bosons. Spinors behave a bit differently from tensors. Better said, the concept of spinors is more related to a continuous topological idea, and not a discontinuous geometrical idea.

To quote Cartan, [39] p.151

> "With the geometric sense we have given to the word spinor it is impossible to introduce fields of spinors into the classical Riemannian technique; that is having chosen an arbitrary system of coordinates $x^i$ for the space, it is impossible to represent a spinor by any finite number N of components, $u_\alpha$, such that the $u_\alpha$ have covariant derivatives of the form
>
> $$u_{\alpha,i} = \partial u_\alpha/\partial x^i + \Lambda^\beta_{\alpha i} u^\beta \qquad (2.179)$$
>
> where the $\Lambda^\beta_{\alpha i}$ are determinate functions of $x^h$."

The problem that Cartan states above has to do with the lack of uniqueness for the covariant transplantation rule when the connection, $\Lambda^\beta_{\alpha i}$, admits affine torsion of the non-integrable variety: $\Lambda^\beta_{\alpha i} - \Lambda^\beta_{i\alpha} \neq 0$. In a Riemannian space with a given metric, the connection coefficients of "parallel" transport are uniquely determined in terms of the Christoffel Symbols. Tensors restricted by neighborhood linearity and the General Linear group admit discrete (discontinuous but unique) transplantation laws about the identity. Spinors, on the other hand, are associated with a certain amount of multi-valuedness, and admit transplantation laws that are continuous about the identity, but are not uniquely defined. When the connection admits affine torsion, there are (at least) two methods of transplantation relating to "right handed" or "left handed" spinors. The multi-valuedness is also a property to be associated with systems that are not *uniquely* integrable in the Frobenius sense, and are characteristic of Huygen envelopes in wave propagation, Cherenkov radiation, and polarization states in Electromagnetism.

The bottom line is that Spinors permit a continuous but not unique evolution about the identity, that is equivalent to the unique but discontinuous linear tensor Vector transformations of negative determinant.

## 2.5 The Hopf Map, Spinors, and Minimal Surfaces

It has been demonstrated in Chapter 1 that there is a close relationship between the Hopf map, minimal surfaces and Spinors. However, the historical lack of reference to these facts indicates that the relationship of Spinor Maps to the Hopf map and minimal surfaces has been ignored by many researchers. Recall that the Hopf map is a (non-linear) map from a vector of 4 components to a vector (the Hopf vector) of 3 components, such that the sum of squares of the three components is the square of the sum of squares of the 4 components. The map is ambiguous to within a sign (plus or minus one). If the components of the Hopf vector in 3 space are presumed to be the dimensionless ratios (x/ct,y/ct/,z/ct), then the Hopf map can be viewed as a map from R4 to a projective 3-space. Fixing the value of the sum of squares of the 4 components to a constant (say unity) generates the equation of the light cone in R4. There are three versions of the (real) Hopf vector, all with the same value for the sum of squares, which can be arranged such that they are mutually orthogonal. The implication is that there are at least three distinct constraints that can represent the light cone.

The correspondence between the Spinor map and the Hopf map will be investigated below, where it will be demonstrated that the rudimentary Cartan definition of a Spinor map is a complex three dimensional "vector" whose real and imaginary components are both Hopf vectors. Each of the two Hopf vectors that make up the Cartan spinor are mutually orthogonal. As mentioned above, it is possible to construct three linearly independent Hopf vectors that are mutually orthogonal, and when these orthogonal Hopf vectors are combined in complex pairs, it is possible to construct six independent spinors. Integration of a given complex null vector leads to a complex "position" vector. The real and imaginary parts of the "position" vector separately describe a pair of (conjugate) minimal surfaces. In differential geometry, the null spinor is called an isotropic vector, and the pair of minimal surfaces are called conjugate surfaces. Linear combinations of the two conjugate components of the "position" vector also generate a minimal surface. The analog in physics can be described in terms of the optics of polarization. One extreme minimal surface is linear polarization, while the other extreme is circular polarization. A linear combination of the minimal surfaces is analogous to elliptical polarization. Each of the polarizations is ambiguous with respect to a sign (right handed vs. left handed, horizontal vs. vertical)

The Hopf map also appears embedded in the classical physics literature. It is latent in the classical optics theory of partial polarization [145]; in the classical electromagnetic theory of Bateman and Whittaker [15]; in the theory of hydrodynamic wakes [195], [197], [194], in the examples of electric wave singular solutions that give the appearance of breaking time reversal symmetry [190]. Yet these classic examples, and many others, do not focus attention on the fact the Hopf vector fields, so constructed in terms of ordered complex pairs, are related to spinors,. It took relativistic quantum theory to focus popularity on spinors, leading to a popular (but false) opinion that spinors were something of a "quantum mechanical"origin. It is now of interest to demonstrate the thermodynamic and cosmological importance of the Hopf map is related to adjoint 1-form (see eq. 1.38 ). The adjoint Hopf 1-form, $A_{Hopf}$, is of Pfaff topological dimension 4, and has a non- zero Topological Torsion vector, $\mathbf{T}_4$, which corresponds to an expansion of space-time. Motion in the direction of $\mathbf{T}_4$ is thermodynamically irreversible. If the expanding universe was modeled in terms of $A_{Hopf}$ then the system would be a turbulent non-equilibrium system of Pfaff dimension 4. However, the evolutionary processes could proceed to domains of Pfaff topological dimension 3, representing condensations, or coherent topological defect structures, (stars - galaxies) that would admit a non-equilibrium, but not dissipative, Hamiltonian evolution, modulo topological fluctuations. For these reasons it is of some importance to study Hopf maps and structures composed of Hopf maps

### *2.5.1 Hopf Maps and Hopf Vectors*

In Chapter 1.2.3, the concept of Hopf vectors was introduced. In this section the complex notation will be used. Consider the map from C2$(\alpha, \beta)$ to R3$(u1, v1, w1)$, as given by the formulas

$$\begin{aligned} \mathbf{H}1 &= [u1, v1, w1] \\ &= [\alpha \cdot \beta^* + \beta \cdot \alpha^*, i(\alpha \cdot \beta^* - \beta \cdot \alpha^*), \alpha \cdot \alpha^* - \beta \cdot \beta^* \end{aligned} \tag{2.180}$$

The variables $\alpha$ and $\beta$ can be viewed also as two distinct complex variables defining ordered pairs of the four variables $[X, Y, Z, S]$. For example, the classic format given above for $\mathbf{H}1$ can be obtained from the expansion,

$$\alpha = X + iY \qquad \beta = Z + iS. \tag{2.181}$$

Other selections for the ordered pairs of $(X, Y, Z, S)$ (along with permutations of the 3 vector components) give distinctly different Hopf vectors. For example, the ordered pairs,

$$\alpha = X + iZ\,, \qquad \beta = Y + iS, \tag{2.182}$$

give

$$\begin{aligned} \mathbf{H}2 &= [u2, v2, w2] \quad (2.183) \\ &= [\alpha\cdot\beta^* + \beta\cdot\alpha^*, \alpha\cdot\alpha^* - \beta\cdot\beta^*, i(\alpha\cdot\beta^* - \beta\cdot\alpha^*)] \\ &= [2(YX - SZ), X^2 + Z^2 - Y^2 - S^2, -2(ZY + SX)] \end{aligned}$$

which is another Hopf vector, a map from R4 to R3, but with the property that **H**2 is orthogonal to **H**1 :

$$\mathbf{H}2 \cdot \mathbf{H}1 = 0. \quad (2.184)$$

Similarly, a third linearly independent orthogonal Hopf vector **H**3 can be found

$$\begin{aligned} \mathbf{H}3 &= [u3, v3, w3] \quad (2.185) \\ &= [\alpha\cdot\alpha^* - \beta\cdot\beta^*, -(\alpha\cdot\beta^* + \beta\cdot\alpha^*), -i(\alpha\cdot\beta^* - \beta\cdot\alpha^*)] \\ &= [X^2 + Y^2 - Z^2 - S^2, -2(YX + SZ), 2(-ZX + SY)] \end{aligned}$$

such that

$$\begin{aligned} \mathbf{H}2\cdot\mathbf{H}1 &= \mathbf{H}3\cdot\mathbf{H}2 = \mathbf{H}2\cdot\mathbf{H}3 = 0. \quad (2.186) \\ \mathbf{H}1\cdot\mathbf{H}1 &= \mathbf{H}2\cdot\mathbf{H}2 = \mathbf{H}3\cdot\mathbf{H}3 = (X^2 + Y^2 + Z^2 + S^2)^2. \quad (2.187) \end{aligned}$$

The three linearly independent Hopf vectors can be used as a basis of R3 excluding the origin. These results are to be compared to Chapter 1.2.3

Each Hopf vector can be differentiated with respect to the variables $(X, Y, Z, S)$ forming a gradient field on R4. That is, the mapping functions $(u, v, w)$ can be differentiated with respect to $(X, Y, Z, S)$ to produce a set of three exact 1- forms. The matrix formed by the three 4 component rows of these gradient fields has an adjoint matrix of coefficients (composed of the matrix of cofactors) which may be adjoined to construct a 4 x 4 basis Frame for R4, excluding the origin. The exterior product, $d(u1)\hat{}d(v1)\hat{}d(w1)$, produces a 3 form, whose components are proportional to those of the adjoint matrix. These components may be used to construct a non-integrable "adjoint" 1-form, $A$. The three exact 1-forms and the non-integrable 1-form also can be used as a basis for the space. The exterior derivatives of the basis frame produce the usual Cartan connection, but the Cartan connection so defined is not free of affine torsion. By this mechanism the differential structure of R4 as induced by the Hopf map is determined.

From another point of view, each of the four functions $X, Y, Z, S$ can be considered as complex variables, so that the Hopf map has a realization from C4 to C3.

### *2.5.2 Isotropic Vectors and Minimal surfaces in 3D*

Along with Cartan, define a rudimentary Spinor as an isotropic (or null) vector of three complex components, $\Sigma = [\sigma 1, \sigma 2, \sigma 3]$ such that

$$(\sigma 1)^2 + (\sigma 2)^2 + (\sigma 3)^2 = 0. \tag{2.188}$$

Next consider the two lemmas given in R. Osserman's book " A Survey of Minimal Surfaces" [146]

> **Lemma 8.1 (Osserman p 63)** *Let D be a domain in the complex z-plane, $g(z)$ an arbitrary meromorphic function in D and $f(z)$ an analytic function in D having the property that at each point where $g(z)$ has a pole of order m, $f(z)$ has a zero of order at least 2m. Then the functions*
>
> $$\begin{aligned} \sigma 1 &= f(1-g^2)/2, & (2.189)\\ \sigma 2 &= i\,f(1+g^2)/2, & (2.190)\\ \sigma 3 &= \mp fg, & (2.191) \end{aligned}$$
>
> *will be analytic in D and satisfy the equation of an "isotropic" null vector:*
>
> $$(\sigma 1)^2 + (\sigma 2)^2 + (\sigma 3)^2 = 0. \tag{2.192}$$
>
> *The only exception is for $\sigma 3 = 0,\ \ \sigma 1 = i\,\sigma 2$.*

Next consider the theorem

> **Lemma 8.2 (Osserman)** E*very simply connected minimal surface in E3 can be represented in the form of a position vector*
>
> $$\mathbf{R}_{\text{real}} = [X1(u, v), Y1(u, v), Z1(u, v)], \tag{2.193}$$
>
> *where $\varpi = (u + i\,v)$. A conjugate minimal surface can be constructed from the imaginary components of the integral formulation,*
>
> $$\mathbf{R}_{\text{imag}} = [X2(u, v), Y2(u, v), Z2(u, v)]. \tag{2.194}$$

*The position vector is computed from an isotropic complex 3 vector by means of the formulas:*

$$X1(u,v) = \operatorname{Re}\int \sigma 1(\varpi)d\varpi + constant \tag{2.195}$$

$$X2(u,v) = \operatorname{Im}\int \sigma 1(\varpi)d\varpi + constant \tag{2.196}$$

$$Y1(u,v) = \operatorname{Re}\int \sigma 2(\varpi)d\varpi + constant \tag{2.197}$$

$$Y2(u,v) = \operatorname{Im}\int \sigma 2(\varpi)d\varpi + constant \tag{2.198}$$

$$Z1(u,v) = \operatorname{Re}\int \sigma 3(\varpi)d\varpi + constant \tag{2.199}$$

$$Z2(u,v) = \operatorname{Im}\int \sigma 3(\varpi)d\varpi + constant \tag{2.200}$$

*Either (real or imaginary) component of the complex position vector, or any linear combination of the components, may be used to induce a 2D real metric, whose Gaussian curvature is negative and whose mean curvatures is zero. Hence it follows that a Cartan Spinor (isotropic 3 vector, Σ) generates (two) minimal surfaces.*

It is unfortunate that the historic word isotropic is used to describe the "null " vector, for in engineering practice, the word isotropic is usually interpreted as meaning the same in all directions. Technically the word isotropic used for the null vector is correct, for no matter what direction the null vector points in C3, its quadratic form, as a sum of squares of the three components, is zero.

An equivalent formulation for an isotropic (null) vector was given by Cartan in terms of $\alpha(z)$ and $\beta(z)$, as follows.

$$\sigma 1 = \alpha^2 - \beta^2, \tag{2.201}$$

$$\sigma 2 = i\,(\alpha^2 + \beta^2), \tag{2.202}$$

$$\sigma 3 = \mp 2\alpha\beta. \tag{2.203}$$

The ambiguity in sign can be related to the concept of polarization.

Evidently D. Sullivan noticed that these formulas of Cartan could be related to minimal surfaces in 1989 (hence predates my own recent independent appreciation (2000) of this fact). The formulas can also be interpreted in terms of the sequence of maps from the 2D space $\{\varpi = u + iv\}$ to the 8D space $\{\alpha = X(\varpi) + iY(\varpi), \beta = Z(\varpi) + iS(\varpi)\}$ to the 6D space $\{\sigma 1(\varpi), \sigma 2(\varpi), \sigma 3(\varpi)\}$. The quadratic form of an arbitrary vector on C3, $(\sigma 1)^2 + (\sigma 2)^2 + (\sigma 3)^2$, can be complex, real, or zero. However, the spinor construction given above always produces an *isotropic* or null vector: the quadratic form vanishes. The mapping described above is the original definition of a Cartan Spinor. A Cartan Spinor is in fact, not the pair of functions, $\alpha(\varpi)$ and $\beta(\varpi)$, but the map to the isotropic complex 3 vector, $\Sigma$, such that

$$(\sigma 1)^2 + (\sigma 2)^2 + (\sigma 3)^2 = 0. \qquad (2.204)$$

The isotropic (null) condition imposes two constraints on the 6D space of 3 complex variables reducing the dimension to a 4D space of two complex variables. In the examples below, for a simple choice of the functions $\alpha$ and $\beta$, the catenoid of revolution occurs as the real part of the integration and the helix is determined from the imaginary part of the integration. A linear combination of the two "conjugate" components is used to form the helicat, which is yet another minimal surface. Each of the functions defined above is ambiguous to a factor of $\pm 1$. The mean curvature vanishes (the minimal surface condition) for all combinations of plus or minus signs.

As mentioned above, the real and imaginary parts of the minimal surface position vector correspond to extremes in "polarization". The interesting fact is that if $\psi$ is a complex "constant" of the type $\psi = A\exp(i\theta)$, then each component of the complex position vector

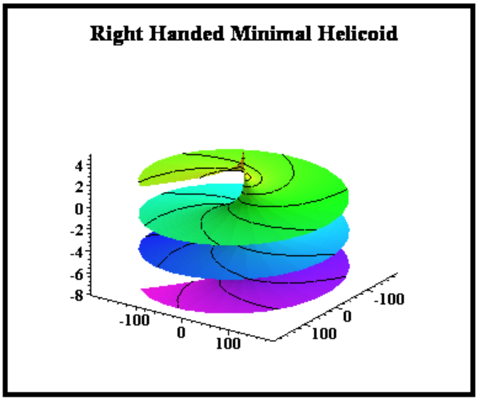
$$\mathbf{R} = A\exp(i\theta)[\mathbf{R}_{\text{real}} + i\mathbf{R}_{\text{imag}}] \qquad (2.205)$$

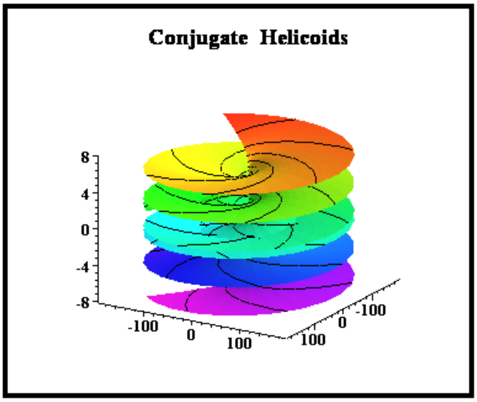
also generates a minimal surface (of mixed polarization)

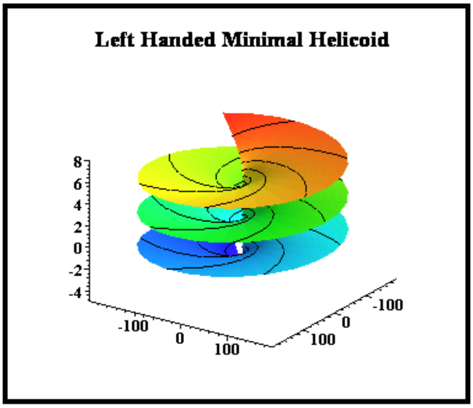
For example consider the position vector in 3 space parameterized by the two variables $u$ and $v$.

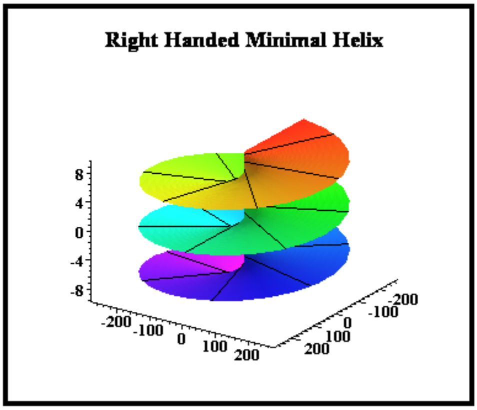
$$\mathbf{R} = [a\ sinh(v)cos(u) - b\ cosh(v)sin(u), \qquad (2.206)$$
$$a\ sinh(v)sin(u) + b\ cosh(v)cos(u), \qquad (2.207)$$
$$au + bv]. \qquad (2.208)$$

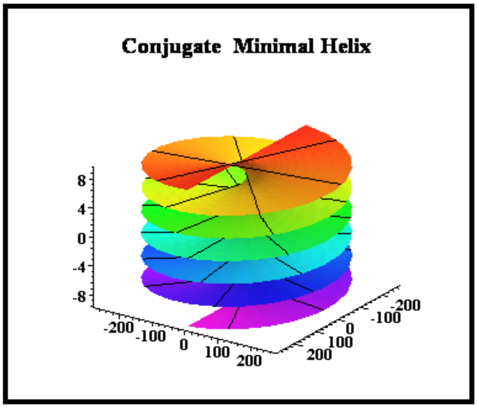
The space curve **R** generates a conjugate pair of minimal surface components for various choices of a and b. For $a = b = 0.5$, the minimal surface consists of two components which are conjugate pairs of opposite handedness. The conjugate pairs can be in a sense "mixtures" of catenoids and helices, and are termed helicoids. The helicoids can also be pure helices or pure catenoids. The situation is remindful of the concept of pure and mixed polarization states in electromagnetic waves. Whether the helicoid is right handed or left handed depends upon whether z is increasing or decreasing. In the first set of three Figures, z is increasing, and $a = b = 0.5$. In the second set of three Figures, $a = 0, b = 1$ and generates the helix structure. In the last set of three Figures, a=1,b=0 which generates the catenoid structure.

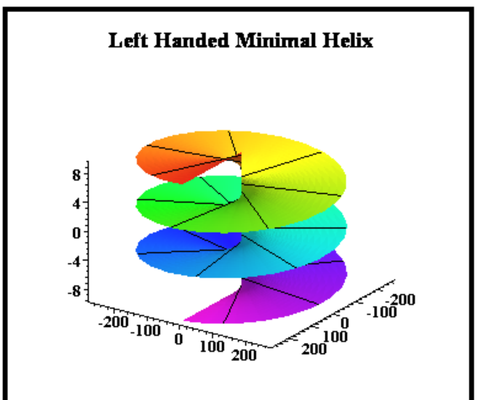

Right Handed Minimal Helicoid
4
2
0
-2
-4
-6
-8
-100
0
100
-100
0
100

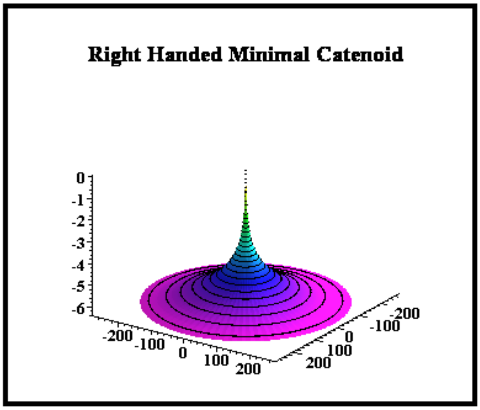

Conjugate Helicoids
8
4
0
-4
-8
-100
0
100
-100
0
100

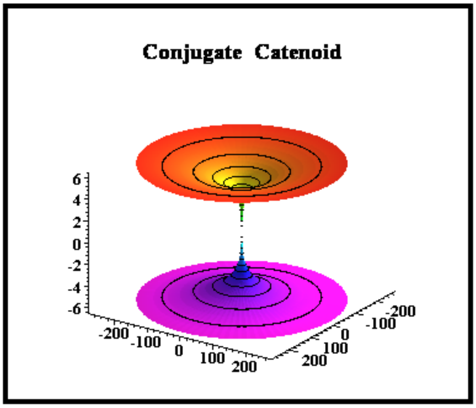

Left Handed Minimal Helicoid
8
6
4
2
0
-2
-4
-100
0
100
-100
0
100

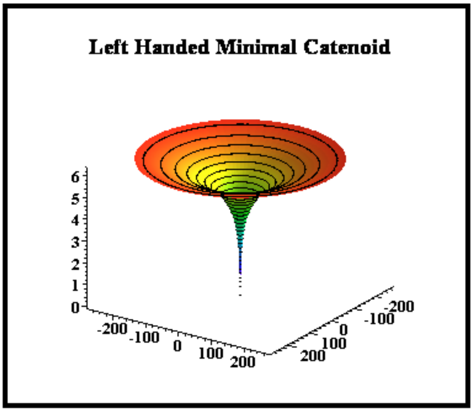

Right Handed Minimal Helix
8
4
0
-4
-8
-200
-100
0
100
200
-200
-100
0
100
200


Conjugate Minimal Helix
8
4
0
-4
-8
-200
-100
0
100
200
-200
-100
0
100
200


Left Handed Minimal Helix
8
4
0
-4
-8
-200
-100
0
100
200
-200
-100
0
100
200


Right Handed Minimal Catenoid
0
-1
-2
-3
-4
-5
-6
-200 -100 0 100 200
-200 -100 0 100 200


Conjugate Catenoid
6
4
2
0
-2
-4
-6
-200 -100 0 100 200
-200 -100 0 100 200


Left Handed Minimal Catenoid
6
5
4
3
2
1
0
-200 -100 0 100 200
-200 -100 0 100 200

It is obvious that this last case of conjugate catenoids has deformable visual features of the Falaco Solitons.

### *2.5.3 Complex Curves*

A theorem of Sophus Lie states that in 4D every complex holomorphic function generates a minimal surface. This is a rather remarkable result that has not been utilized fully in application to understanding space-time evolutionary processes. If an evolutionary process starts with a non-holomorphic representation and evolves into a holomorphic representation, then physically it would be expected that dissipative processes would be minimized, and tangential discontinuities (wakes) would be created. The normal field to a minimal surface is harmonic.

For Navier-Stokes like fluids, the viscous shear dissipation is a viscous coefficient times the vector Laplacian of the the velocity field. As the vector Laplacian vanishes for a harmonic vector field it follows that such flows do not experience viscous dissipation due to shears. Consider vector fields that are composed of a harmonic part and a non-harmonic part. Viscous dissipation will cause the non-harmonic part to decay. What is left is the Harmonic part, which generates a minimal surface as the (measurable) wake.

Consider a complex curve defined in terms of $\xi = u + iv$ as

$$R = [\xi, \Theta(\xi)] = [u + iv, \mathrm{Re}(\Theta(\xi) + i\,\mathrm{Im}(\Theta(\xi))] \tag{2.209}$$

$$\Rightarrow [u, v, \Phi, \Psi]. \tag{2.210}$$

Osserman (p. 19 [146]) demonstrates how this 4 dimensional position vector satifys the minimal surface equation. The Minimal surface generated by a complex curve, does not admit a single implicit real function in 3D for its description. Such minimal surfaces are artifacts of 4D space time.

Consider an evolutionary system like a fluid in space time. Consider a complex holomorphic curve. This complex curve induces a two dimension subspace of space time. The subspace is a minimal surface. However, there does exist a parametrization of such a minimal surface, and that is what the Weierstrass method is all about. The method represents a parametric version, 2D into 3D, but not implicit version, 3D to a constant. The "parametric" vector to the surface can be used to describe a vorticity field (or a velocity field). Such a vector is harmonic.

For thermodynamic systems that can be encoded by a 1-form of Action, of Pfaff Topological dimension 4 (the canonical example is the Hopf map), the Universal Phase function is a holomorphic function of the complex eigenvalue $\xi$

$$\begin{aligned}
\Theta(x,y,z,t;\ \xi) &= \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi + T_K \Rightarrow 0, && (2.211)\\
\mathrm{Re}(\Theta(z)) &= u^4 - 6u^2v^2 + v^4 - X_M(u^2 - 3v^2)u && (2.212)\\
&+ Y_G(u^2 - v^2) - Z_A u + T_K && (2.213)\\
\mathrm{Im}(\Theta(\xi) &= 4(u^2 - v^2)uv - X_M(3u^2 - v^2)v + 2Y_G uv - Z_A v && (2.214)
\end{aligned}$$

Hence the Universal Phase function defines a complex curve in terms of the similarity coefficients. A cosmological universe of Pfaff topological dimension 4 can be put into correspondence with cojugate mimimal surfaces.

### *2.5.4 Spinors and the Hopf map*

The isotropic Complex position vector, $[z1, z2, z3]$ can be decomposed into a real and imaginary part, such that both have the same sum of squares, and are orthogonal. In other words, the Cartan Spinor can be represented as

$$|\boldsymbol{\sigma}_{12}\rangle = |\mathbf{H}1\rangle + i\ |\mathbf{H}2\rangle \quad \textit{with} \quad \langle\boldsymbol{\sigma}_{12}| \circ |\boldsymbol{\sigma}_{12}\rangle = 0 \tag{2.215}$$

Two other Cartan spinors are represented by the combinations.

$$\begin{aligned}
|\boldsymbol{\sigma}_{23}\rangle &= |\mathbf{H}2\rangle + i\ |\mathbf{H}3\rangle \quad \textit{with} \quad \langle\boldsymbol{\sigma}_{23}| \circ |\boldsymbol{\sigma}_{23}\rangle = 0, && (2.216)\\
|\boldsymbol{\sigma}_{31}\rangle &= |\mathbf{H}3\rangle + i\ |\mathbf{H}1\rangle \quad \textit{with} \quad \langle\boldsymbol{\sigma}_{31}| \circ |\boldsymbol{\sigma}_{31}\rangle = 0 && (2.217)
\end{aligned}$$

These formulas can be obtained from the Cartan representation for the isotropic 3 vector. As an example consider the permuted form,

$$\begin{aligned}
z1 &= \alpha^2 - \beta^2, && (2.218)\\
z2 &= -2\alpha\beta, && (2.219)\\
z3 &= i\,(\alpha^2 + \beta^2). && (2.220)
\end{aligned}$$

Make the substitutions $\{\alpha = X + iZ, \quad \beta = Y - iS\}$ to obtain the equations

$$|\boldsymbol{\sigma}_{31}\rangle = \left|\begin{matrix} X^2 + S^2 - Y^2 - Z^2 + i\,2(ZX + SY) \\ -2(YX + SZ) + i\,2(-ZY + SX) \\ 2(-ZX + SY) + i\,(X^2 + Y^2 - Z^2 - S^2) \end{matrix}\right\rangle = |\mathbf{H}3\rangle + i\ |\mathbf{H}1\rangle \tag{2.221}$$

### 2.5.5 The Adjoint field to the Hopf Map

The Hopf Map, as characterized by the equations:

$$[u1, v1, w1] = [2(XZ + YS), 2(XS - YZ), (X^2 + Y^2) - (Z^2 + S^2)], \quad (2.222)$$

can be used to generated 3 linear independent 1-forms on R4, by forming the gradient with respect to $[X, Y, Z, S]$ of each of the three functions that define the map. These three covariant 4 component vectors may be used in the construction of a frame matrix on R4. A fourth linearly independent vector is needed, to complete the basis frame. This fourth vector can be constructed from the adjoint operation (on matrices or differential forms) to within an arbitrary scaling factor,$1/\lambda$. The linearly independent 1-forms are therefor,

$$d(u1) = 2Zd(X) + 2Sd(Y) + 2Xd(Z) + 2Yd(S), \quad (2.223)$$
$$d(v1) = 2Sd(X) - 2Zd(Y) - 2Yd(Z) + 2Xd(S), \quad (2.224)$$
$$d(w1) = 2Xd(X) + 2Yd(Y) - 2Zd(Z) - Sd(S), \quad (2.225)$$

$$A_{Hopf} = \{-Yd(X) + Xd(Y) - Sd(Z) + Zd(S)\}/\Lambda. \quad (2.226)$$

The Frame Matrix so generated is given by the expression:

$$F = \begin{bmatrix} Z & S & X & Y \\ S & -Z & -Y & X \\ X & Y & -Z & -S \\ -Y/\lambda & X/\lambda & -S/\lambda & Z/\lambda \end{bmatrix}, \quad Det[F] = \left(Z^2 + S^2 + Y^2 + X^2\right)^2/\lambda \quad (2.227)$$

It is some interest to examine the properties of the adjoint 1-form, $A_{Hopf}$,defined hereafter as the Hopf 1-form. For $\lambda = 1$, it follows that the Hopf 1-form is of Pfaff dimension 4.

It is also of interest to consider factors $\lambda$ that are of the format of the Holder norm, where n and p are integers, and (a,b,k,m) are arbitrary constants.

$$\lambda = (aX^p + bY^p + kZ^p + mS^p)^{n/p} \quad (2.228)$$

The exponents n and p determine the homogeneity of the resulting 1-form, which is given below an ambiguous format (the plus of minus sign)

$$A_{\pm} = A_{\pm}/\lambda = \{\pm(Yd(X) - Xd(Y)) - Sd(Z) + Zd(S)\}/\lambda. \qquad (2.229)$$

For example, for n=p=2, the scaling factor becomes related to the classic quadratic form. The scaled Hopf 1-form, $A$, is then homogeneous of degree zero.

For arbitrary n and p, the 3-form of topological (Hopf) torsion

$$\text{Topological Torsion} \qquad (2.230)$$
$$= (A_{\pm})\hat{}d(A_{\pm}) = i(\pm\mathbf{T}_4)d(X)\hat{}d(Y)\hat{}d(Z)\hat{}d(S) \qquad (2.231)$$

generates a direction field defined as the 4 component Torsion vector, $\mathbf{T}_4$.

$$\mathbf{T}_4 = \pm[X, Y, Z, S]/\lambda. \qquad (2.232)$$

The factor $\beta$ depends upon the integers n and p as well as the constants ( a,b,k,m).

The Topological Parity 4-form, whose coefficient is the 4 divergence of the Torsion vector, $\mathbf{T}_4$, becomes

$$\text{Topological Parity } d(A_{\pm})\hat{}(d(A_{\pm}) \qquad (2.233)$$
$$= -4(\pm\lambda)^{(-2n/p)})(n-2)d(X)\hat{}d(Y)\hat{}d(Z)\hat{}d(S) \qquad (2.234)$$

It is most remarkable that for n=2, any p and any (a,b,k,m), the topological parity vanishes and the scaled Hopf 1-form is of Pfaff dimension 3, not 4. In such cases the ratios of the integrals of the topological torsion 3 form over various closed manifolds are rational, and the closed integrals of the 3-form are topological deformation invariants. (coherent structures).

Also note that if the scaling factor is restricted to values such that $n = 4,\ p = 2,\ a = b = k = m = 1,$ then the Frame matrix is unimodular, and the scaled Hopf 1-form is homogeneous of degree -2, relative to the substitution $X \Rightarrow \gamma X$, *etc.* (A somewhat different definition of homogeneity relative to the volume element will be given below.) For this constraint, the 2-form, $F = dA$, has two components in analog to the **E** and **B** fields of electromagnetism. The two 3 component "blades" are identical only when all of the coefficients are equal to unity. A finite value for the quadratic form leads to a sphere in 3D of coordinates u1,u2,u3.

**Electromagnetism of Index zero Hopf 1-forms**

Guided by prior investigations, it is of interest to use the scaled Hopf 1-form as the generator of electromagnetic field intensities. The coefficients of the scaled Hopf 1-form can be put into correspondence with the classic vector and scalar potentials, $[\mathbf{A}, \phi]$ (using $S = CT$). The Action for the first examples is then of the format,

$$A_{\pm,0} = A_{\pm}/\lambda_0 = \{\pm(+Yd(X) - Xd(Y)) - CTd(Z) + CZd(T)\}/\lambda_0 \quad (2.235)$$

When the number of minus signs in the quadratic form is zero (index 0), and the exponents are n=4, p=2, such that

$$\lambda_0 = (X^2 + Y^2 + Z^2 + S^2)^2, \quad (2.236)$$

then it is remarkable that the derived 2-form has coefficients (**E** and **B**) that are proportional to the same Hopf Map with the the classic result that $\mathbf{E}^2 = C^2\mathbf{B}^2$, Using the minus ambiguity (parity) sign, the **E** field is anti-parallel to the **B** field. If the positive ambiguity (parity) sign is used, the **E** and **B** fields are parallel:

$$F = dA, \quad (2.237)$$
$$\mathbf{B} = curl\mathbf{A} = \quad (2.238)$$
$$[2(CTY + XZ),$$
$$-2(-YZ + CTX), \quad (2.239)$$
$$(-X^2 - Y^2 + Z^2 + (CT)^2)](2/(\lambda_0)^{3/2}) \quad (2.240)$$
$$\mathbf{E} = -grad\phi - \partial\mathbf{A}/\partial T = \quad (2.241)$$
$$[-2(CTY + XZ),$$
$$2(-YZ + CTX), \quad (2.242)$$
$$-(-X^2 - Y^2 + Z^2 + (CT)^2)](2C/(\lambda_0)^{3/2}) \quad (2.243)$$

It is natural to ask if these **E** and **B** fields admit a Lorentz symmetry constitutive constraint such that vacuum state is charge current free. Recall that a constitutive constraint is a relationship between a 2-form, $F$, and a 2-form density $G$, such that the coefficients of $G(\mathbf{D}, \mathbf{H})$ are related to the coefficients of $F(\mathbf{E}, \mathbf{B})$. A Lorenz vacuum condition implies that the fields are solutions of the vector wave equation. The question becomes, "If is presumed that $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, do the Maxwell Ampere equations generate a zero 3 form of charge current? ". Direct computation of the index zero Hopf 1-form indicates that $dG = J \neq 0$, unless $\varepsilon\mu C^2 + 1 = 0$. Hence the scaled Hopf Action, where the scaling is of signature zero, does **not** describe a charge current free vacuum, for real positive values of $\varepsilon$, $\mu$, and $C$. On the other hand, if it is presumed that the domain is such that say $\mu$,or $\varepsilon$, is negative, then the Hopf Map, scaled as above, does generate charge-current free wave solutions. Negative $\varepsilon$ appears to hold in metals and the Earth's ionosphere. Recent announcements indicate constructions that yield negative $\mu$ [151]. However, for situations where $\varepsilon$ or $\mu$ are negative, the Hopf wave solutions imply that the Spin angular momentum $A\hat{}G$ is not a deformation invariant (hence Spin angular momentum of the field is not conserved.)

### Electromagnetism of Index one Hopf 1-forms

When the number of minus signs in the quadratic form is one (index 1), and the exponents are n=4, p=2, such that (using lower case letters for Index one Hopf 1-forms)

$$\lambda_1 = (x^2 + y^2 + z^2 - c^2t^2)^2, \tag{2.244}$$

then it is remarkable that the derived 2-form has coefficients ($\mathbf{E}$ and $\mathbf{B}$) that are proportional to different Hopf Maps. The Action 1-form is the same as above, but with a different denominator.

$$A_{\pm,1} = A_{(\pm)}/\lambda_1 = \{\pm(yd(x) - xd(y)) - Ctd(z) + zCd(t)\}/\lambda_1 \tag{2.245}$$

The fact leads to the classic result that $\mathbf{E}^2 = C^2\mathbf{B}^2$, but now the $\mathbf{E}$ field is not collinear with the $\mathbf{B}$ field. Using the negative ambiguity (parity) sign leads to the fields:

$$F = dA \quad (2.246)$$
$$\mathbf{B} = curl\mathbf{A} = \quad (2.247)$$
$$[2(Cty + xz),$$
$$-2(-yz + Ctx), \quad (2.248)$$
$$-x^2 - y^2 + z^2 + (Ct)^2)](2/(\lambda_1)^{3/2}) \quad (2.249)$$
$$\mathbf{E} = -grad\phi - \partial\mathbf{A}/\partial t = \quad (2.250)$$
$$[2(Cty - xz),$$
$$2(-yz - Ctx), \quad (2.251)$$
$$-(-x^2 - y^2 + z^2 + (Ct)^2)](2C/(\lambda_1)^{3/2}) \quad (2.252)$$

Independent from any other constraints, it is possible to construct the 3-form of Topological Torsion, and its exterior derivative defined as Topological Parity. The Topological parity can be either positive, zero, or negative. For the example Hopf 1-form given above (using the negative ambiguity sign), the Topological Torsion is represented to within a factor by a position vector$[-x, -y, -z, -t]$ inbound in 4 dimensions, and having a negative divergence or parity. If the positive sign of the ambiguity factor is changed, then the parity of the form changes sign. For example, for the 1-form,

$$A1 = A1_+/\lambda_1 = \{+yd(x) - xd(y) - Ctd(z) + zCd(t)\}/\lambda_1, \quad (2.253)$$

the 4 -form of topological parity is positive, and the topological torsion is represented by an outbound position vector (to within a factor).

Similar to the investigation described above, it is natural to ask if these **E** and **B** fields admit a Lorentz symmetry constitutive constraint such that vacuum state is charge current free. Again, such a condition implies that the fields are solutions of the vector wave equation. Direct computation of the Maxwell Ampere equations indicates that $dG = J = 0$ if the phase velocity constraint vanishes, $\varepsilon\mu C^2 - 1 = 0$. Hence the scaled Hopf Action, where the scaling is of index one, **does** describe a charge current free vacuum, for real positive values of $\varepsilon$, $\mu$, and $C$.

It is some interest to give the charge current solutions to show how the "phase factor " $(\varepsilon\mu C^2 - 1) \Rightarrow 0$ establishes the vacuum charge free conditions.

$$J^x = -(yx^2 + yz^2 + 5yC^2t^2 - 6zCtx + y^3)(\varepsilon\mu C^2 - 1)4/\lambda^2 \quad (2.254)$$

$$J^y = (x^3 + xy^2 + xz^2 + 5xC^2t^2 + 6zCty)(\varepsilon\mu C^2 - 1)4/\lambda^2 \tag{2.255}$$

$$J^z = -(2x^2 + 2y^2 - z^2 + C^2t^2)(\varepsilon\mu C^2 - 1)8Ct/\lambda^2 \tag{2.256}$$

$$\rho = 0 \tag{2.257}$$

Note that there are possible charge current free (wave solutions) that are governed by curves in space time generated by the intersection of the three surfaces created by setting the coefficients of the current density equal to zero. These solutions are valid for any phase velocity.

The given solution above is not free of Topological Torsion, $A\hat{}F$, and there is a non-zero value of the second Poincare invariant, $\mathbf{E}\cdot\mathbf{B} \neq 0$. However, the Spin 3-form $A\hat{}G$ is also non-zero [164] [207], but it has, subject to the phase constraint, a zero 4-divergence. (The first Poincare invariant is zero.) The divergence of the Spin 3-form, has 2 parts. The first part is twice the conventional Lagrange density of the fields, $(\mathbf{B}\cdot\mathbf{H} - \mathbf{D}\cdot\mathbf{E})$. The second part is the interaction between the potentials and the charge currents, $(\mathbf{A}\cdot\mathbf{J} - \rho\phi)$. When the divergence of the 3-form is zero, then the closed integrals of Topological Spin are deformation invariants, and have closed integrals with rational (quantized) ratios. That is, with regard to any singly parametrized vector field, $V$, describing an evolutionary process,

$$\begin{aligned} L_{(\beta V)} \int_{z3} (A\hat{}G) &= \int_{z3} i(\beta V)d(A\hat{}G) + \int_{z3} d(i(\beta V)A\hat{}G) \quad (2.258) \\ &= 0 + 0 \supset \textit{evolutionary invariance.} \end{aligned}$$

The function $\beta$ is an arbitrary deformation parameter.

### Twistors composed by superposing two index 1 Hopf 1-forms

By superposing (adding or subtracting) two different, index 1, Hopf 1-forms (which will be shown below to be equivalent to a Penrose twistor solution) it is possible to construct a vacuum (charge current free wave) solution to the Maxwell system, subject to the constraint that the phase speed satisfies the phase velocity equation, $(\varepsilon\mu C^2 - 1) = 0$.

As an example consider another Hopf 1-form of index 1 formulated as

$$A2 = A2_+/\lambda_1 = \{Ctd(x) + zd(y) - yd(z) - xCd(t)\}/\lambda_1 \tag{2.259}$$

Similar formulas for the field intensities can be determined as above. Note that the parity of the Hopf forms to be superposed can be the same or different. If the parity of the two superposed Hopf 1-forms are opposite, then without consideration of the phase constraint, the Topological Torsion of the "twistor" 1-form vanishes, $A\hat{}F = 0$. Yet the quantized topological spin3-form $A\hat{}G$ does not vanish, and moreover, subject to the phase constraint, the closed integrals of the Spin 3 form are conserved. This result implies that such a construction yields "quantized" values for the Spin integrals.

In this "twistor" case, note the vector represented by the vector $R = [x, y, z, t]$ in R4, is orthogonal to the 1-form of Action. It follows that for a twistor Action,

$$A = A1_- + A2_+ \tag{2.260}$$

$$i(R)A = 0,\ and\ L_{(R)}A = 2A \tag{2.261}$$

$$i(R)dA = 2A,\ and\ L_{(R)}dA = 2dA \tag{2.262}$$

Note that the Hopf 1 form, $A$ and the derived 2-form, $F = dA$, are both homogeneous of degree 2, with respect to $R$.

The "twistor" Action created by superposing Hopf 1-forms of different parity (but not the general Hopf action) is integrable in the sense of Frobenius,

$$\text{Topological Torsion } H = A\hat{}F = 0. \tag{2.263}$$

The implication is that the 4 forms of Topological Parity, or the second Poincare invariant, (which does not depend upon constitutive properties) is also zero for the twistor 1-form:

$$Second\,Poincare\,invariant\,PII \tag{2.264}$$

$$= d(A\hat{}F) = F\hat{}F = 2\mathbf{E}\cdot\mathbf{B}dx\hat{}dy\hat{}dz\hat{}dt \Rightarrow 0 \tag{2.265}$$

Classically, one would say that the second Poincare invariant vanishes for this twistor Action.

From the constitutive relations, there exists a 3-form (density) $S$, (Kiehn 1976) defined as the Spin 3-form,

$$S := A\hat{}G \text{ such that } A\hat{}S = 0. \tag{2.266}$$

The Action of the Lie Derivative on the Spin 1-form, $S$, is such that

$$L_{(R)}S = (L_{(R)}A)\hat{}G + A\hat{}L_{(R)}G = 2A\hat{}G + A\hat{}(2G) = 4S \tag{2.267}$$

and

$$L_{(R)}dS = 4dS. \tag{2.268}$$

The Spin 3-form, $S$, and its divergence 4 form, $dS$ are homogeneous of degree 4 relative to the vector $R$. Subject to the phase constraint, the divergence of the Spin 3-form vanishes, which indicates that the closed integrals of the spin 3-form are conserved as period integrals.

These results are to be compared with the Penrose twistor definitions in terms of differential forms [150] The energy flow $\mathbf{E} \times \mathbf{H}$ of such a solution is collinear with the spatial components of the Spin, $\mathbf{S}$.

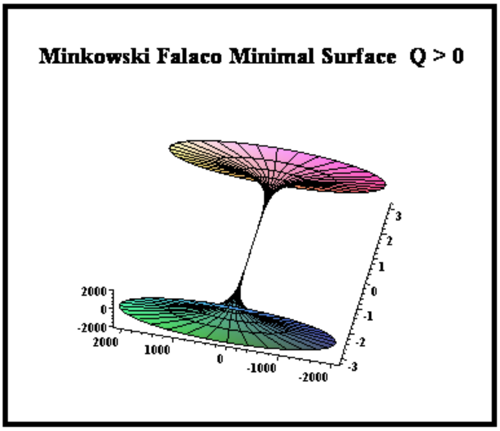
## 2.6 Interesting Cosmological Conjectures

### *2.6.1 Minimal Surfaces in Minkowski space*

Most of the discussion about minimal surfaces that appears in the current literature is dominated by the assumption of a Euclidean metric. However, it is possible to discuss surfaces in Minkowski spaces of 3 and 4 dimensions. For example consider the position vector in 3D given by the parametric expression in Minkowski space with signature $[-1, 1, 1]$

$$R = [v, (1/a)sinh(av + b)cos(u), (1/a)sinh(av + b)sin(u)] \tag{2.269}$$

Then straight forward computations for the parametric surface leads to the conclusion that the mean curvature is zero (hence the surface is a minimal surface), but has a Gauss curvature which is positive. Recall that in Euclidean space, a minimal surface has a negative Gauss curvature, which makes these results strange. However, the plot of the Minkowski space minimal surface is not so strange, and reminiscent of the Falaco Solitons.




*2.6.2 Point Particles as Real and Complex Spheres of "zero radii"*

A point particle is typically modeled as a 3 dimensional euclidean real ball with a vanishingly small radius vector. The length of the radius vector squared is defined by the sum of squares of its real components. The surface area of the real ball tends to zero as the length of the radius shrinks. However if a "point" particle is defined as a (complex) sphere of vanishingly small radius, then complex point particles could be represented by an isotropic null vector, whose length squared is defined, in the same euclidean manner as for real vectors, as the sum of the squares of its components. In a Euclidean space (where the signature of the fundamental quadratic form is zero) the isotropic vector is not realized in terms of real variables. In Minkowski space, where the signature of the fundamental form is 1, the isotropic vectors (of null length) can be represented by real vectors, relative to the pseudometric. It is suggested herein that a physical "point" in real Euclidean space be extended to include complex euclidean space, and/or Minkowski space. The surface area of a real "point" is zero in real euclidean space, but the surface area of a complex "point" can be finite, even though its "diameter" is zero. This result follows from the fact that an isotropic null vector can be used as the generator of a minimal surface (see the subsection on "Isotropic Vectors and Minimal surfaces in 3D"). The minimal surface is not of zero area. The idea is to study a "point" volume of "zero" real radius that is bounded by two minimal (perhaps conjugate) minimal surfaces. The concept of a spinor is interpreted as the topological realization of a "point" particle of finite area, but zero diameter.

### *2.6.3 More on Minimal Surfaces*

It is extraordinary that the Hopf Adjoint vector, when suitably normalized to have coefficients homogeneous of degree zero, can be used to define a minimal surface in 3D, where the Gauss curvature (sums of product pairs of curvatures) is real and positive. Real minimal surfaces in 3D have a Gauss curvature (sums of product pairs of curvatures) which is negative.

Consider the Hopf Adjoint vector is of the form

$$A_0 = b(ydx - xdy) + a(tdz - zdt). \tag{2.270}$$

The 1-form of Potentials depends on the coefficients $a$ and $b$ which are presumed to take on values $\pm 1$. There are two cases corresponding to left and right handed "polarizations": $a = b$ or $a = -b$. (There actually are 6 cases to consider, by cyclically permuting the variables, and these can be combined to represent spinor solutions.)

Next normalize the 1-form by dividing through by a Holder norm such that the coefficients of the renormalized 1-form are homogeneous of degree zero. Then construct the similarity invariants of the Jacobian matrix determined from the coefficients of the renormalized 1-form. What is remarkable for this example, is that both the Mean curvature (sum of curvatures), the Adjoint (Cubic curvature = sum of all curvature triples), and quadratic curvature (determinant of the Jacobian matrix = product of all curvatures) of the implicit hypersurface in 4D vanish, for any choice of a or b. The Gauss curvature (sum of all pairs of curvatures) is non-zero, positive, real and is equal to the square of the radius of a 4D euclidean sphere. The cubic interaction energy density is zero.

$$Mean = 0, \tag{2.271}$$
$$Gauss > 0 \tag{2.272}$$
$$Cubic = 0 \tag{2.273}$$
$$Top\_Torsion \neq 0 \tag{2.274}$$
$$J_s \neq 0 \tag{2.275}$$

This situation occurs when the three curvatures of the implicit 3-surface are $\{0, +i\omega, -i\omega)$.. This Hopf surface is therefore a 3D imaginary *minimal* two dimensional hyper surface in 4D and has two non-zero pure imaginary curvatures! Strangely enough the charge-current density is not zero, but it is proportional to the topological Torsion vector that generates the 3 form $A\hat{}F$. The topological Parity 4 form is not zero, and depends on the sign of the coefficients a and b. In other words the 'handedness' of the different 1-forms determines the orientation of the normal field with respect to the implicit surface.

It is also possible to deduce a closed 3-form of "Charge-Current density", $J_s$, for such 3D hypersurfaces. The coefficients of $A\hat{}J_s$ are exactly equal to the "Cubic" curvature similarity invariant. The spatial scalar product of $A$ and $J$ is balanced by the product $\rho\phi$. It is known that a process described by a vector proportional to the topological torsion vector in a domain where the topological parity $(4ba)$ is non-zero is thermodynamically irreversible.

It is also possible to construct combinations of chirally different Hopf Adjoint 1-forms to find what are called Instanton solutions [143] [73].

### *2.6.4 Bulk Viscosity and Cosmology*

A Google search (October 2004) yields over 5000 articles that utilize the concepts of Bulk Viscosity in a General Relatistic treatment of dark matter, and almost 3,000,000 articles on dark matter and dark energy. In all cases the theories are more or less phenomenological. It is apparent that the authors do not realize that Bulk Viscosity is a topological effect related to the divergence of the Topological Torsion tensor and the expansion - contraction or rotational shears that may occur in a 4D space time variety. For a cosmological universe encoded in terms of a 1-form of Action of Pfaff dimension 4, the Bulk Viscosity coefficient is proportional (in EM notation) to $\mathbf{E} \circ \mathbf{B}$ and the dissipation of irreversible processes depends upon parity of contraction or expansion, or the chirality of rotation. Dark matter is an artifact of irreversible topological evolution.

### *2.6.5 The Four Forces and Differential Topology*

Almost twenty years ago [168], an argument was presented to show how the properties of the four forces in physics could be deduced from the features of the four distinct Pfaffian equivalence classes of differential geometry that can be constructed on a space of four dimensions. The four equivalence classes were determined from the metric solutions, $g_{\mu\nu}$ , to the Einstein field equations, by constructing a 1-form of action, $A$, in terms of the space time, $g_{4\nu}$, components of the metric field: $A = g_{4\nu}dx^{\nu}$. The methods of Pfaff reduction can be used to generate four equivalence classes in terms of the Pfaff Topological dimension, or class, of this 1-form. Summarizing the previous results, the methods lead to:

1. (Newtonian Force) The equivalence class of Pfaff Topological dimen-

sion 1 will support long range gravitation (mass) and is parity preserving.

2. (Coulomb Force) The second equivalence class of Pfaff Topological dimension 2 will support both gravity (mass) and electromagnetism (charge) and is to be associated with long range parity preserving forces.
3. (Strong Force) The third equivalence class of Pfaff Topological dimension 3 will support both mass and charge, but the forces - although parity preserving– are of short range.
4. (Weak Force) The last equivalence class of Pfaff Topological dimension 4 involved short range interactions that can violate time reversal and symmetry breaking .

Examples were given in terms of known solutions to the Einstein field equations.

| Solution to Einstein equations | Pfaff Topological dimension of $A = g_{4\nu} dx^{\nu}$ |
|---|---|
| Schwarzschild | 1 |
| Riessner-Nordstrom | 2 |
| Godel | 3 |
| Kerr Taub Nut | 4 |

Although the previous methods were motivated by ideas of differential geometry, it is now known that the concepts used to generate the four equivalence classes associated with the four forces are not of a geometrical nature, but instead are better expressed in terms of equivalence classes which have their foundations in the topological property of Pfaff Topological dimension. Indeed, the older analysis concluded that two of the equivalence classes are to be associated with forces that are long range, in the sense of having "distance" limits that extended to infinity The other two equivalence classes are to be associated with forces that are of short range. However, the concept of "distance" is more of a geometrical idea, not a topological idea.

Now it is perceived that the true nature of the equivalence classes is based on the topological issue of connectedness, and does not reflect the geometrical idea of distance necessarily. Following the work of Baldwin (see Vol. 1, Chapter 6), two of the equivalence classes belong to a connected topology (Pfaff Topological dimension 1 and 2), and the other two equivalence classes belong to a disconnected topology (Pfaff dimension 3 and 4). Hence the topological features of the strong and the weak forces do not involve short range, but instead reflect the concepts of accessibility. That is, the Cartan topology of the "long range" forces is connected, while the Cartan topology of the "short range" forces is disconnected. The topological idea of connectedness is to be exchanged for the geometrical idea of "long range or distance". There is a difference between the concepts of whether or not, from point a, the point b is not "reachable" by a continuous process and not "reachable" in a finite time. Moreover, the Pfaff topological dimension can be associated with Thermodynamic features. Systems of Pfaff Topological dimension greater than 2 are not in thermodynamic equilibrium.

These ideas are most readily understood in terms of the Cartan topology built on a Pfaffian system, and its differential closure. These sets have global properties, and therefore carry topological significance. These concepts of Pfaff equivalence classes have application not only to the microcosm of atoms and elementary particles, as well as the cosmological arena of galaxies, but also to the mundane physics of hydrodynamics. Such methods have been used recently to obtain a better understanding of the production of wake patterns, and the creation and decay of turbulence in fluids.

### *2.6.6 Signature Symmetry Breaking*

However, over the years a new feature of the analysis has appeared, and it is to this new feature that this section is directed. Note that in the 1975 reference, the signature of the quadratic form was taken to be { +,+,+,- }. The question now arises: Is there a symmetry to be broken if one considers the often used but opposite signature { -,-,-,+ }. The idea is that the wave equation

$$+\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 + \partial^2\psi/\partial z^2 = +(1/c^2)\partial^2\psi/\partial t^2, \tag{2.276}$$

has a set of characteristics which satisfy the partial differential system:

$$+(\partial\psi/\partial x)^2 + (\partial\psi/\partial y)^2 + (\partial\psi/\partial z)^2 = +(1/c^2)(\partial\psi/\partial t)^2. \tag{2.277}$$

Hence, there are two ways to write this constraint as an algebraic variety ( a null set) :

$$+(\partial\psi/\partial x)^2 + (\partial\psi/\partial y)^2 + (\partial\psi/\partial z)^2 - (1/c^2)(\partial\psi/\partial t)^2 = 0, \tag{2.278}$$

or

$$-(\partial\psi/\partial x)^2 - (\partial\psi/\partial y)^2 - (\partial\psi/\partial z)^2 + (1/c^2)(\partial\psi/\partial t)^2. \qquad (2.279)$$

Each quadratic form is the complete mirror symmetry (the negative) of the other, but it turns out that the signatures are intrinsically different from a topological point of view in the neighborhood of the null variety.

The analytic question that remains is: Does this symmetry of space time signatures have distinguishable consequences? The physical question is: Are there experiments that can be done to distinguish the symmetry breaking between {-,-,-,+} and {+,+,+,-}?

The analytic answer, based on the idea that the Clifford Algebras of such systems are not isomorphic to one another [12], is yes! The mathematical argument is similar to that used to distinguish the two species of angular momentum algebras in quantum mechanics, an argument which is based on the different signatures of the raising or lowering operators (commutator or anti-commutator brackets) for Bosons vs. Fermions. The fact that the differences in angular momentum signature are physically observable implies that the differences in space-time signatures may also be measurable.

Consider the Clifford Algebra with signature {+,+,+,-}. As discussed in reference [12], this algebra is isomorphic to the algebra of 4x4 matrices with real numbers as matrix elements. This matrix algebra is the usual representation used for waves in 4 dimensions. Next consider the Clifford Algebra with signature {-,-,-,+}. This algebra is isomorphic to the algebra of 2x2 matrices with quaternions as matrix elements. The non-abelian quality of the quaternions makes this algebra have extraordinary differences from the algebra of 4x4 matrices over the real numbers.

This positive analytic result which breaks the symmetry between the two space-time signatures implies there must be a physical difference between the two types of space-time, one with signature {+,+,+,-}, and the other with signature {-,-,-,+}. These differences imply that there exist two species of waves. What are they? A possible answer was first given by Schultz [175] who found exact quaternionic solutions to Maxwell's equations that indicated that the speed of propagation in the inbound and outbound directions would be different for such waves. This result was in agreement with the ring laser experiments of Sanders [174]. These sets of experiments indicated that the electromagnetic four fold degeneracy of the Lorentz equivalence class could be broken such that all four waves of left - right polarizations and of to - fro propagation directions would propagate at four distinct speeds. A further more general analysis on the macroscopic parity and time reversal symmetry breaking effects in electromagnetic systems was presented in reference [190]. The question of whether or not these waves, or the effects of {+,+,+,-} vs. {-,-,-,+} signatures, produce any quantum or hydromechanical effects is open.

### *2.6.7 Hedgehog fields, Rotating plasmas, Accretion discs*

Using Maple (see Vol. 6, "Maple programs for non Equilibrium systems") , it is possible to find a modification of a closed 1-form solution to Maxwell's equations that makes the magnetic field lines appear like the spines of a Hedgehog. It is also possible to demonstrate how such modifications of closed 1-forms make the z=0 plane of a rotating plasma a chiral attractor. Consider

$$\mathbf{A} = \Gamma(x,y,z,t)[-y,x,0]/(x^2+y^2)\ , \tag{2.280}$$

$$\textit{with}\quad \Gamma = -z\,m/\sqrt{(x^2+y^2+\epsilon z^2)} \tag{2.281}$$

$$\textit{and}\quad \phi = 0. \tag{2.282}$$

These potentials induce the field intensities:

$$\mathbf{E} = [0,0,0], \tag{2.283}$$

$$\mathbf{B} = m\,[x,y,z]/(x^2+y^2+\epsilon\,z^2)^{3/2}. \tag{2.284}$$

The **B** field is of the format of the famous Dirac Hedgehog field often associated with "magnetic monopoles". However, the radial **B** field has zero divergence everywhere except at the origin, which herein is interpreted as a topological obstruction. The factor $\epsilon$ is to be interpreted as an oblateness factor associated with rotation of a plasma, and is a number between zero and 1. It is apparent that the helicity density and the second Poincare invariant are zero:

$$\mathbf{E}\circ\mathbf{B} = 0 \qquad \textit{and} \qquad \mathbf{A}\circ\mathbf{B} = 0\,. \tag{2.285}$$

In fact, the 3-form of topological torsion vanishes identically (as $\phi = 0$),

$$\mathbf{T}_4 = [0,0,0,0]. \tag{2.286}$$

In this example, there is a non zero value for the Amperian current density, even though the potentials are static. The Current Density 3-form has components,

$$\mathbf{J}_4 = (3m/2\mu)\ (1-\epsilon)\,z\,[-y,x,0,0]/(x^2+y^2+\epsilon\,z^2)^{5/2}.. \tag{2.287}$$

which do not vanish if the system is "oblate" ($0 < \epsilon < 1$). This current density has a sense of "circulation" about the z axis, and is proportional to the vector potential reminiscent of a London current, $\mathbf{J} = \lambda\mathbf{A}$. The "order" parameter is $(3/2\mu)\ (1-\epsilon)/(x^2+y^2+\epsilon\, z^2)^2$.

The Lorentz force can be computed as:

$$\mathbf{J}\times\mathbf{B} = (3m^2/4\mu)\ (1-\epsilon)[xz^2, yz^2, -z]/(x^2+y^2+\epsilon\, z^2)^2 \qquad (2.288)$$

The formula demonstrates that the Lorentz force on the plasma, for the given system of circulating currents, is directed radially away (centrifugally) from the rotational axis, and yet is such that the plasma is attracted to the $z = 0, xy$ plane. The Lorentz force is divergent in the radial plane and convergent in the direction of the z axis, towards the z=0 plane. This electromagnetic field, therefor, would have the tendency to form an accretion disk of the plasma in the presence of a central gravitational field.

Although the 3-form of Topological Torsion vanishes identically, the 3-form of Spin is not zero. The spatial components of the Spin are opposite to the direction field of the Lorentz force (in the sense of a radiation reaction).

$$\mathbf{S}_4 = (m^2/4\mu)[xz^2, yz^2, -z, 0]/(x^2+y^2+\epsilon\, z^2)^2. \qquad (2.289)$$

The components of the Spin 3-form are in fact proportional to the components of the virtual work 1-form with the ratio $-3(1-\epsilon)$ depending on the oblateness factor.

It is also true that the divergence of the 3-form of spin is not zero, for the first Poincare invariant is

$$d(A\hat{}G) \Rightarrow P1 = (m^2/4\mu)(x^2+y^2+4(1-\epsilon)\, z^2)/(x^2+y^2+\epsilon\, z^2)^3 \quad (2.290)$$

For a more detailed discussion see Vol. 4 "Plasmas and Non equilibrium Electrodynamics" [250].

### *2.6.8 Chandrasekhar Black Holes and the Hopf Map*

All known Black Hole solutions to the cosmology of General Relativity are Petrov type D solutions [42]. Petrov type D solutions are analogous to Universal Phase functions with conjugate pairs of equal roots. It has been demonstrated above that the Hopf map produces similar qualities, and generates a minimal surface as well. Another feature of the Black Hole solutions is the fact that $\mathbf{E}\circ\mathbf{B} \neq \mathbf{0}$ [54]. Such is also the case with the Hopf 1-form of Action. Super position of Hopf 1-forms can be used to generate Instantons on a space of 4 topological dimensions. The fundamental idea is that these cosmological systems are non-equilibrium thermodynamic systems of Pfaff Topological Dimension 4.

As an example, consider the composite 1-form of Action

$$A_c = \{(ydx - xdy) + (sdz - zds)\} \quad (2.291)$$
$$+\{(xdz - zdx) + (sdy - yds)\} \quad (2.292)$$
$$+\{(zdy - ydz) + (sdx - xds)\}, \quad (2.293)$$

which is a combination of 3 Hopf adjoint 1-forms. The derived 2-form is equal to

$$F = dA = 2(dy\hat{}dx + dz\hat{}dy + dx\hat{}dz) + 2(ds\hat{}dz + ds\hat{}dx + ds\hat{}dy). \quad (2.294)$$

and

$$F\hat{}F = 10(dx\hat{}dy\hat{}dz\hat{}ds). \quad (2.295)$$

If $\lambda$ is defined as

$$\lambda = (x^2 + y^2 + z^2 + s^2)^{1/2}, \quad (2.296)$$

and if the composite 1-form, $A_c$, is rescaled by the factor

$$f = 2/(\lambda^2 + c^2) \quad (2.297)$$

then the result is the "Instanton Potential" (pp169-170 [73])

$$\text{Instanton 1-form} = 2A_c/(\lambda^2 + c^2). \quad (2.298)$$

If the coefficient c = 0, then the 3-form of topological torsion constructed from the rescaled 1-form has zero divergence. Hence it is closed, but not exact 3-form, and by deRham's theorems produces quantized values when integrated over closed 3 D cycles.

### *2.6.9 Dark matter, Dark energy (Negative Pressure), Energy Balance and Curvatures.*

Astronomical measurements over the last 20 years have been interpreted to imply that the constituents of our Universe are ordinary matter (~5%), dark matter (25%) and dark energy (70%). Statements that dark matter and dark energy compose more that 95 % of the energy of the universe have been quite surprising. As mentioned above a Google search gives over 3 million articles on dark matter and dark energy.

In terms of a metrically based gravitational theory, the presence of dark matter has been inferred from the observed dynamics of cosmic objects, particularly from fast rotation of hydrogen clouds far outside the luminous disc of spiral galaxies, as well as high-velocity dispersion of galaxies in clusters. Very little (at present) is known about dark energy and dark matter, even though it would seem from the recent conjectures and interpretations that more than 95% of the universe is these forms. A rash of publications involving quantum virtual states and interactions coupled with general relativity concepts have been offered to "explain" the interpretations of astronomical data. Could there be a more mundane theory that would offer less "far out" explanations?

That being said, That being said, dark energy has the following defining properties:

1. Dark energy emits no light;
2. Dark energy it has large, negative pressure, $p = -\rho c^2$
3. Dark energy it is approximately homogeneous.

Apparently dark energy does not cluster significantly with matter on scales at least as large as clusters of galaxies. Because its pressure is comparable in magnitude to its energy density, it is more "energy-like" than "matter-like" (matter being characterized by $p << \rho c^2$ ). Dark energy is qualitatively very different from dark matter. The three forms can be summarily classified by how their energy densities change with a cosmic scale factor a: ordinary and dark matter behave as $a^{-3}$, radiation, which interacts with ordinary matter, behaves as $a^{-4}$, while dark energy, at least in the simplest models thereof, is independent of the scale, a.

The ordinary matter content can be deduced from observations at e.g. visible or radio frequencies. The presence of dark matter can be inferred from the observed dynamics of cosmic objects, particularly from fast rotation of hydrogen clouds far outside the luminous disc of spiral galaxies, as well as high-velocity dispersion of galaxies in clusters. Dark energy, or quintessence, is equivalent to a non-zero cosmological constant, , in Einstein's equations, and there is recent support for a non-zero also from redshift observations. All of these direct measurements can be compared with theoretical cosmology and the observed angular structure of the CMB. The theorists claim that a sequence of peaks should indeed arise from coherent acoustic oscillations in the baryon-photon fluid during an early epoch. Their amplitudes and relative positions provide another series of tests of cosmological models, and put a different series of constraints on the parameters of such models,

The attempts to explain the fundamental "cause" of such surprising results (that 95% of cosmological matter is unknown) seem to verge on the ludicrous. Perhaps a review of some older physical ideas may be of value. In particular, the theory of real gases, as modeled by the universal van der Waals gas in space time, is a theory that can be expressed not only in terms of Gauss curvature (a quadratic curvature property of implicit surfaces) but also in terms of molar interaction and cohesion Cubic curvature effects ( Pressures which can be both positive and negative - see A. Sommerfeld), and linear Mean curvature effects (that dominate surface tension and string theories).

None of these concepts (similarly to most thermodynamic concepts) depend upon size – hence metric is not explicitly required. Curvatures are generated from the similarity invariants of a 4th order Cayley Hamilton polynomial equation, deducible from the Jacobian matrix of the 1-form of Action used to encode a specific physical system. Yet there is a fundamental universal equation that links all of the different "energy - curvature" terms.

From a topological point of view:

1. Pressure is related to volume (a "3D" thing)

2. Temperature is related to Entropy (a "2D" thing)

3. Tension is related to a length (a "1D" thing)

and all of these terms appear in the equations describing a van der Waals gas. The concepts are universal in that they are deformation invariants. The universal Phase function can always be deformed into a van der Walls gas. There is always a critical point, a spinodal line of ultimate stability (a winged cusp, or swallowtail bifurcation), a binodal line (Pitchfork bifurcation) that defines phase transitions, a line of critical temperature (Hysteretic bifurcation) that separates gas from vapor and liquid condensates. None of what has been said so far requires metric, or connection, or gauge theory.

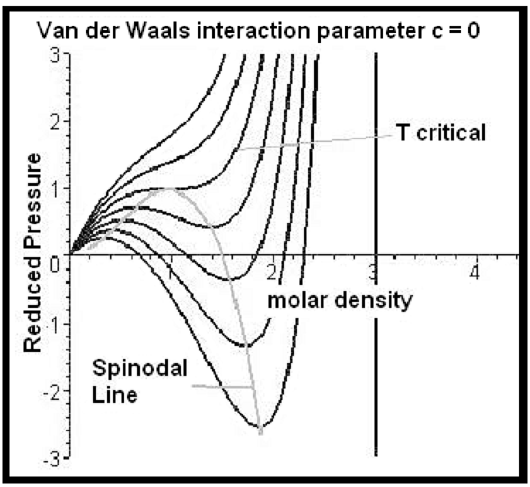
From experiment one should expect wild fluctuations on molar density near the critical point. By comparison to the van der Waals gas, the eigen values, $\xi$, play the role of molar density. The Mean curvature $M$ is related to a "length" as a conjugate variable, and is a measure of the "surface tension". The Gauss curvature $G$ has an "area" as a conjugate variable, and is related to temperature. The cubic curvature $A$ has a "volume" as a conjugate variable, and is related to Pressure. The "energy balance" equation (to within a factor) becomes a sum of 4 parts

1- String_tension + Entropy - Pressure, (2.299)

1- Mean curvature + Gauss curvature - Cubic curvature (2.300)

Nowhere has there been imposed a restriction on negative values of Pressure, or Temperature, or Surface tension. The divergence =0 condition (Ricci curvature) is automatic, for Pfaff dimension 4 conditions.

For example, the Pressure - molar density diagram for the van der Waals gas is repeated below:




**Note the region of possible negative pressure**.

Note the region of "negative" pressure for the van der Waals gas, an effect know to most steam engine designers. No mention of vacuum fluctuations, or quantum effects, has been made.

Could it not be possible that dark matter is an effect due to bulk viscosity of a topologically 4 dimensional physical system (the universe) evolving with a fixed point of expansion or rotation, and that dark energy is merely negative pressures associated with a non-equilibrium van der Waals gas?

# Chapter 3
# THE ARROW OF TIME

## 3.1 Retrodiction vs. Prediction

The idea that a dissipative process is associated with a change in topology (see Vol 1.) implies that non-adiabatic, radiative transitions *cannot* be described by a homeomorphism*. Accordingly, the transformational behavior of physical fields with respect to continuous, but not necessarily homeomorphic, maps is of some interest to researchers who study non-adiabatic transitions. For tensor fields and configurations, the classical transformation rules can, and must, be modified slightly to produce a useful concept of natural, or intrinsic, covariance with respect to maps which may not even have invertible Jacobians. Surprisingly enough, the techniques presented below indicate that for covariant tensor fields there exists a natural sense of retrodictive determinism which does not exist for contravariant tensor fields. Moreover, the logical structure is not symmetric, nor even dualistic, with the ideas of prediction of tensor fields. It appears that a system described by a tensor field may be predictively statistical, but retrodictively deterministic.

A major purpose of this section is to sharpen the perception of researchers to the transformational asymmetries associated with tensor fields, such that these ideas may be fruitfully exploited and incorporated into physical theories. A table summarizing the transformation properties of tensor rules w.r.t. maps of varying degrees of invertibility and differentiability is presented, along with a few abstract applications directed towards subtle points in the theory of thermodynamics and hydrodynamics.

### *3.1.1 Intrinsic Covariance and the Classification of Maps*

The basic ideas to be utilized are straightforward, but appear only sporadically, if at all, in the engineering literature . (For a partial, mathematical

*Much of this material was developed in 1976 [172].

treatment, see [116]) The definition of a contravariant tensor is taken to be the classic one which uses the transformation rule,

$$X^A \to Y^\mu = J^\mu_A X^A \tag{3.1}$$

to define contravariant quantities. The Jacobian matrix, $J^\mu_A(X)$, is constructed in terms of the partial derivatives, $\partial\phi^\mu/\partial x^A$ of the map $\phi : x^A \to y^\mu$, all of which are assumed to exist, but which need not be continuous.

However, a covariant tensor is defined herein to be an object which obeys the transformation rule,

$$Y_\mu J^\mu_K = X_K \leftarrow Y_\mu \tag{3.2}$$

which, contrary to the usual tensor analysis definition, does not make use of a Jacobian inverse (the base spaces need not be of the same dimension). These two rules may be used to develop the ideas of natural, or intrinsic, covariance w.r.t. maps for which the Jacobian inverse does not (globally) exist, with the ideas agreeing with classic results when the transformations are restricted to orthogonal transformations or diffeomorphisms.

Scattered through the literature of differential topology are discussions of various maps which may or may not have inverses and whose Jacobians may or may not be invertible. A summary of such maps is given in Table 1, where more importantly, the intrinsic covariant behavior of tensor fields with respect to each class of maps is also presented. A tensor field is considered as a set of functional rules over a domain, with values in some range. The intrinsic covariance problem considered herein (and with results presented in Table 1) relates to the global solubility, and uniqueness of solubility, of the *rules* (not just their values) w.r.t. various transformations of the base spaces, as domains. For example: given two domains, x and y, with a map $\phi$ between them from x to y, is it possible to uniquely determine a tensor rule over the final state in terms of a tensor rule given over the initial state? The answer is no, if the map $\phi$ does not have an inverse. Surprisingly enough, a retrodictive version of this question obtains a favorable response for covariant tensor fields: Is it possible to retrodictively determine a co-tensor rule over an initial state, given a co-tensor rule over a final state? An affirmative answer requires only that the Jacobian coefficients of map $\phi : x^A \to y^\mu$ exist.

| Map | Existence Properties | Co-rule | Contra-rule |
|---|---|---|---|
| Continuous | $\phi, d\phi$ | $R$ | - |
| Submersion | $\phi, d\phi_{onto}$ | $R_{unique}$ | - |
| Immersion | $\phi, d\phi_{1-1}$ | $R$ | $R_{unique}$ |
| Local Inverse | $\phi, d\phi_{onto}, d\phi_{1-1}$ | $R_{unique}$ | $R_{unique}$ |
| Cont, disc. Inverse | $\phi, \phi^{-1}, d\phi$ | $R$ | $P$ |
| Disc, Cont Inverse | $\phi, \phi^{-1}, d\phi^{-1}$ | $P$ | $R$ |
| Submanifold | $\phi, \phi^{-1}, d\phi_{1-1}$ | $R$ | $(R\,\&\,P)_{uniq.}$ |
| Quotient manifold | $\phi, \phi^{-1}, d\phi_{onto}$ | $(R\,\&\,P)_{uniq}$ | $P$ |
| Homeomorphisms | $\phi, \phi^{-1}, d\phi, d\phi^{-1}$ | $R\,\&\,P$ | $R\,\&\,P$ |
| Embedding | $\phi, \phi^{-1}, d\phi_{1-1}, d\phi^{-1}_{onto}$ | $R\,\&\,P$ | $(R\,\&\,P)_{uniq.}$ |
| Projection | $\phi, \phi^{-1}, d\phi_{onto}, d\phi^{-1}_{1-1}$ | $(R\,\&\,P)_{uniq}$ | $R\,\&\,P$ |
| Diffeomorphism | $d\phi\,\&\,d\phi^{-1}$ 1-1 & onto | $(R\,\&\,P)_{uniq}$ | $(R\,\&\,P)_{uniq.}$ |

Table 1. Retrodictive and predictive behavior of tensor fields w.r.t. continuous maps
with different invertibility and differential structure.
$R =$ Retrodictive, $P =$Predictive

A summary of the transformational solubility (and uniqueness of solutions) of tensor fields with respect to various maps is presented in Table 1. Certain details are enumerated in Appendix A.

The logical asymmetry exhibited by the table is remarkable, as is the fact that for non-homeomorphic maps (which are necessary to represent dissipative transitions) there exists the possibility of a retrodictive determinism – but not a predictive determinism. There appears to be an arrow of time built into the transformational behavior of tensor fields with respect to non-homeomorphic maps. A recognition of this built in logical asymmetry should be taken into account by those theories which treat irreversible processes.

### *3.1.2 Physical Applications*

There seems to exist a predilection in physics to obtain a description of nature which is predictive. The classic problem in point mechanics is a problem in prediction: given initial data, what is the future trajectory? Watanabe states that "Every closed system of physical laws must include a time-dependent law from which it is possible to deduce a predictive statement..." and ". . . physical theory is preeminently a predictive instrument.."[256] [257]. Now it is not apparent to this author that predictive descriptions are the only way to understand nature, and a study of the transformation properties of tensor fields given in Table I gives support to the position that although a deterministic, predictive, analysis for dissipative systems is impossible, surprisingly often the opposite point of view, based on a deterministic *retrodictive* analysis, is possible for dissipative systems. There is a definite asymmetry in dissipative processes and according to Table I this asymmetry persists in the analytical description of such systems.

The map classification table is striking in that it points out and emphasizes the natural retrodictive logical structure for tensor fields, especially covariant fields, with respect to maps that are not homeomorphisms. As is emphasized in the non-equilibrium theory of continuous topological involution, irreversible processes imply changing topology. For such systems, Table 1 indicates that a retrodictive analysis is appropriate, and moreover, it is the only analysis that is well defined in a functional sense, if the process is irreversible ($\phi^{-1}$ does not exist). Perhaps the fundamental reason that Cartan's theory of differential forms, built on alternating covariant tensor fields, is so powerful is due to the retrodictive solubility of differential forms with respect to $C^1$ maps. It has been shown (see Vol. 1) that the specification of a system of differential forms is equivalent to specifying a topology and the utilization of Pfaffian expressions (which are differential forms of degree one) in the science of thermodynamics was an early recognition of the need for injection of topological concepts into thermodynamic theories. Caratheodory's use of neighborhoods [95], Landsberg's use of restricted continuity and the frontier of a set [115] and the modern work of Boyling [23] are examples of the utility of a topological approach in thermodynamics. It is in the study of Non Equilibrium Systems and Irreversible Processes that Cartan's mathematics of differential forms demonstrates its power, for according to the previous discussion, differential forms are well behaved (at least retrodictably) with respect to non-homeomorphic, topology-changing maps. Cartan's theory of exterior differential forms appears to be the appropriate mathematical theory for studying dissipative systems.

For physical applications the two most important principle maps are the immersion and the submersion. Ordinarily, these maps are to be used in the field sense. That is, the physical object is considered to be the base space which is immersed or submersed into a Euclidean tensor space. The submersion induces on the physical object, as a manifold, a set of covariant vector lines which form an orthogonal field (on the object manifold) w.r.t. the fibers of the submersion. The "gradient" vector to the spherical surface $\phi : R^3 \to R = \{x^2 + y^2 + z^2 = 1.0\}$ is an example; the fiber is the spherical surface itself. The orthogonal field spans the compliment to the fiber space created by the submersion.

On the other hand, the immersion of the object manifold into a Euclidean space, induces a global covariant metric field, $g_{\mu\nu}$, on the base space, which permits a norm to be created for contravariant vectors on the manifold. The idea of distance along a line is well defined by the immersion. The notion of distance between "surfaces" may not be well defined by an immersion (consider the birefringent crystal); this concept requires a reciprocal metric field that the object manifold may not support, globally.

For physical objects which are manifolds there exist theorems that imply that they always may be immersed into a Euclidean space of sufficiently large dimension [226]. The implication is that manifolds always support a global metric field, whose covariant columns form a global, linearly independent set of differential forms. The determinant of the induced metric field is never zero, functionally, but can take on both positive and negative values, *discontinuously.* However, if the induced map, $d\phi$, of the immersion is continuous, as well as being 1-1, then the determinant of the metric field is never zero and must be definite.

In a topological sense, those manifolds which are immersed in a continuously differentiable manner must be orientable. A non-orientable manifold cannot support a covariant metric field with definite determinant. This subtle point is at the basis of Caratheodory's proof of the second part of the second law of thermodynamics [33]. Entropy is positive definite only on orientable manifolds. A set of points, or states, whose topology excludes, or makes inaccessible, another set of points, or states, supports a monotonic function only if the topology is orientable. A Mobius strip is a model of a topology (infinitely extendable) which produces inaccessible states, but one for which the entropy function is not globally definite (S $>$ 0). Therefore, an immediate application of the point of view discussed above is to demonstrate a subtle and usually not expressed assumption in the theory of thermodynamics: The phase space of Gibbs must be an *orientable* sub-manifold of state space, if entropy is definite.

For contravariant considerations of a physical object as a manifold, the next most important maps are the Submanifold map and the Quotient manifold map. These mappings in physical situations are usually *from* a Euclidean space to the body manifold are made in such a way as to permit contravariant fields to be induced on the body manifold. (The submersion and immersion described above were *from* the body manifold to the Euclidean space, and induced covariant, not contravariant, fields on M) The classic example is the submanifold map, $\phi$, which carries the unit interval into a curve in M. The induced differential map, $d\phi$, defines a tangent (contravariant) vector on M, in the sense of Lagrange, which spans the submanifold of M. This notion is distinct from the submersive (Hamiltonian) case which defines a covariant wave vector field on M.

If the map to M is a quotient manifold map, then a reciprocal metric field is induced on M which permits a distance between "surfaces" concept to be defined globally. Dual to the immersive case, a distance between "points" may not be admissible. The idea is that in the immersive case a contravariant measure is induced; in the quotient manifold case, a covariant measure is induced. The measure *coefficients* are covariant and contravariant, respectively, for the above mentioned measures. The two measure fields have different transformation properties for non homeomorphic dynamical transformations. Physically, the notion of strain is related to the covariant measure coefficients, while the notion of stress is related to the contravariant measure coefficients.

These results emphasize the differences between Lagrangian (contravariant particle) and Hamiltonian (covariant wave) mechanics [165] — differences which become evident only for dissipative systems that do not admit global metrics and reciprocal metrics. For the dissipative case, there must exist two sets of physical laws: one for the covariant ideas, one for the contravariant ideas. The differential form statements for Maxwell's equations are the foremost example of such dual behavior. The first Maxwell pair of equations involving Faraday's law and covariant **E** and **B** intensities is one statement. The distinctly different second Maxwell pair of equations involves contravariant quantities, **D** and **H** and is the second statement. There is a fundamental difference between physical *intensities*, such as the (covariant) **E** and **B** fields of electromagnetism, when compared to physical *quantities* such as the (contravariant densities) **D** and **H** fields [155][225] . The differences are degenerate unless the system is irreversible, a fact that implies that all physical phenomena which can be deduced from the behavior of **E** and **B** fields can be metrically deduced from the behavior of **D** and **H** fields, in non-dissipative systems. For a space that does not support both a global metric field and a reciprocal metric field, one set of equations does not uniquely imply the other. (The dual to the Einstein equation for the covariant metric field is unknown.)

A study of these results should guide the development of physical theories for dissipative systems. Such theories, without a dynamical inverse, are not amenable to predictive determinism. These systems involve changing topology, but nevertheless, such dynamical systems (if describable by continuous maps) will yield a retrodictive determinism but only for a covariant (wave) formalism. Contravariant (particle-trajectory) formalisms are never predictive if $\phi^{-1}$ does not exist. The physicist, for dissipative field problems, should adopt the view: Given the final data, what was the initial state from which it came? (This statement is dual, but not reciprocal to the usual Cauchy statement: given initial data, what is the final state? Curiously enough, this point of view seems to have been taken by Hadamard [79]). Such questions and their answers, although not predictive in style, also yield an understanding of nature. Moreover the methods are deterministic, not statistical and are employed in a retrodictive sense.

Perhaps the most obvious physical example of a continuously dissipative system is the turbulent fluid. The deterministic theory of a turbulent fluid has yet to be formulated, apparently because of a predilection for a predictive theory. Moreover, from the arguments given above, as the dissipative turbulent flow does not admit an inverse, a predictive deterministic theory in terms of velocity fields is impossible. Again, the point of view discussed above has led to an immediate application by proving once and for all that a predictive, non-statistical theory of turbulence is impossible. Since the time of G. I. Taylor, turbulence theories that have made any progress have been predictively statistical and non-deterministic. However, the alternate point of view, based upon the deterministic retrodiction of differential forms, is just beginning to be utilized. Early results of the theory have demonstrated that (1) if the system is dissipative, topology must change and (2) a turbulent system *cannot* be described by Hamilton's equations of motion; i.e., a Hamiltonian analysis of a turbulent fluid is impossible. Moreover, if a flow is to be diffusively dissipative (an intuitive requirement of turbulent flow) then the Liouville theorem must fail. The notion that the Liouville theorem must fail disallows the utilization of a symplectic geometry approach (better said: symplectic transformations cannot be utilized to study turbulence, but there exist uniquely defined thermodynamic irreversible processes on the symplectic manifold! [1]). The topological criteria imply that not only are groups not admissible ($\phi^{-1}$ does not exist), but also semi-groups are not admissible in a turbulent flow. Also it has been demonstrated that the Navier-Stokes equations for a viscous fluid cannot be derived from a strictly Hamiltonian analysis, but indeed are representable by the covariant concepts embodied in the theory of differential forms. The purpose of this article is to focus attention on the logical basis of the statement that, indeed, the physics of fields is deterministically a retrodictive science. The permissibility of physics being deterministically predictive is not the usual case and demands the special constraints of a non-dissipative system.

### *3.1.3 Transformational covariance*

To demonstrate the transformation asymmetry of tensor fields this subsection considers maps between spaces of different dimensionality, $\phi$" $M \rightarrow N$, from points $x^A$ in the domain to points $y^\mu$ in the range. It is assumed that a physical system can be described as a tensor field, i.e. by a map $\alpha : M \rightarrow \tau^A$ in the initial state and a map $\overline{\alpha} : N \rightarrow \overline{\tau}^\mu$ in the final state. Both of these maps are from the base space (M or N) to the contravariant tensor space ($\tau^A$ or $\overline{\tau}^\mu$ ). An alternative field description can be made in terms of the maps $\beta : M \rightarrow \tau_K$ and $\overline{\beta} : N \rightarrow \tau_\upsilon$, which are from the base spaces to the covariant tensor spaces. The contravariant and covariant fields behave differently with respect to predictive and retrodictive deterministic solubility and part of the purpose of this subsection is to demonstrate these differences in behavior, *even though the field values may be related by a metric.*

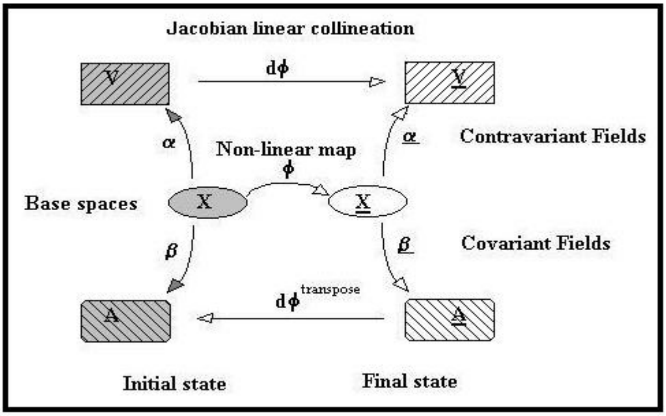
For purposes herein it will be assumed that $\phi$ exists andthe Jacobian matrix of partial derivatives $\partial\phi^\mu/\partial x^A = J^\mu_A$ exists; i.e., unless otherwise specified, the map $\phi$ is continuous. The Jacobian matrix induces two linear maps, $d\phi : \tau^A \rightarrow \overline{\tau}^\mu$ and $\widetilde{d\phi} : \tau_K \leftarrow \tau_\upsilon$ between the tensor spaces. The direction of the arrows is important; they demonstrate that the Jacobian matrix always permits the values of the contravariant fields, as numbers, to be predicted (but not necessarily retrodicted) and similarly the covariant values may be retrodicted w.r.t. $\phi$ (but not necessarily predicted). The usual rules are expressible in coordinate language as,

$$d\phi : V^A(x) \rightarrow V^\mu(x) = \sum_A J^\mu_A(x) V^A(x) \tag{A1}$$

and

$$\widetilde{d\phi} : A_K(x) = \sum_\mu A_\mu(y(x)) J^\mu_K(x) \leftarrow A_\mu(y) \tag{A2}$$

Note that the existence of a Jacobian inverse has not been assumed. The action of the Jacobian is to push forward the values of contravariant fields and "pull back" the values of the covariant fields. A diagrammatic description is given in Figure 1. Classical developments of tensor analysis assume that an inverse Jacobian matrix exists such that the rule for covariant tensor transformation instead of being given by (A2) becomes $[d\phi]^{-1} : A_K(x) \rightarrow A_\mu(x) = [J^\mu_K(x)]^{-1} = A_K(x)$. This transition rule will not be assumed herein.




**Induced maps between tensor fields.**

Equations (Al) and (A2) not only yield recipes for computing values of covariant and contravariant tensors, but also explicitly demonstrate the differences between covariant and contravariant fields. Equation (A2) permits the covariant rule, $\beta$, to be deduced retrodictively from the covariant rule, $\overline{\beta}$. On the other hand, equation (Al) for contravariant values does not permit the contravariant rule, $\overline{\alpha}$, to be deduced predictively from the rule $\alpha$, for the functions $V^{\mu}(x)$ have arguments on the domain space, $x$ and are not functions of variables, $y$, on the range space. This fact is the fundamental observation which distinguishes between covariant/contravariant and retrodictive/predictive analysis. Covariant fields (especially differential forms) are always retrodictive, even for irreversible maps where $\phi^{-1}$ does not exist. This principle result is easily deduced from the directionality of the arrows in Figure above. A study of the arrows in the figure will lead to a quick understanding of the transformational behavior for the tensor field rules, $\alpha$ and $\beta$, as more invertibility, or differentiability, structure is assumed for the map $\phi$.

As an example of the analysis, consider the submersion for which $d\phi$ is onto ($\widetilde{d\phi}$ 1-1). For submersions, $d\phi$ admits a right inverse ($J \circ \widetilde{J}$ is non-singular on the range space) and by modifying the arrows in Fig. I it is easy to see that the co-values are both predictive and retrodictive, the contra-values are predictive and the covariant rule $\beta$ is retrodictively *unique.* (The proof can be effected easily by writing in a bijective arrow for $\widetilde{d\phi}$ .)

If the map $\phi$ is an immersion such that $d\phi$ is 1-1 ($\widetilde{d\phi}$ is onto), then the map permits a left inverse of $d\phi$ ($\widetilde{J} \circ J$ is non-singular on the domain space).

Co-values and covariant rules are retrodictive. Contra-values are both predictive and retrodictive, but now the contravariant rules are retrodictively *unique.*

If the map $\phi$ is such that $d\phi$ is both 1-1 and onto, then co- and contra-values are both retrodictive and predictive and both the contravariant rule and covariant rule are retrodictively *unique.*

None of the above maps admit an inverse function, globally; none of the rules are predictive. The "arrow of time" permits determinism only in a retrodictive sense.

A continuous map, with a discontinuous inverse, yields enough additional structure beyond the primitive cases considered above such that for the first time the contravariant rule, $\alpha$, is predictive. A completely dual situation occurs for those discontinuous maps $\phi$ that admit an inverse $\phi^{-1}$ and $d\phi^{-1}$ (but no $d\phi$): the co-values and covariant rules become predictive and the contra-values and contravariant rules become retrodictive.

Of somewhat greater importance to physical systems are the maximal rank Submanifold map, for which $\phi$, $\phi^{-1}$' and $d\phi_{1-1}$, are valid globally, and its dual, which is the Quotient manifold map, for which $\phi$, $\phi^{-1}$and $d\phi_{onto}$ are globally valid. These maps permit contravariant vector fields to be induced globally on a manifold.

None of the maps considered so far are homeomorphisms; they do not necessarily preserve topology. Subsequent maps to be considered are homeomorphisms and all admit continuous inverses. Dissipative, irreversible systems cannot be described by such maps. Dissipative systems imply a change in topology.

For the weakest homeomorphism, $\phi$ and $d\phi$ exist and similarly $\phi^{-1}$ and $d\phi^{-1}$ exist. For such maps, both contravariant values and rules, as well as covariant values and rules, are soluble in both a retrodictive and predictive manner.

If the homeomorphism is an embedding, $d\phi$ is 1-1 and $d\phi^{-1}$ is onto; the contravariant rules become uniquely soluble in both a predictive and a retrodictive sense.

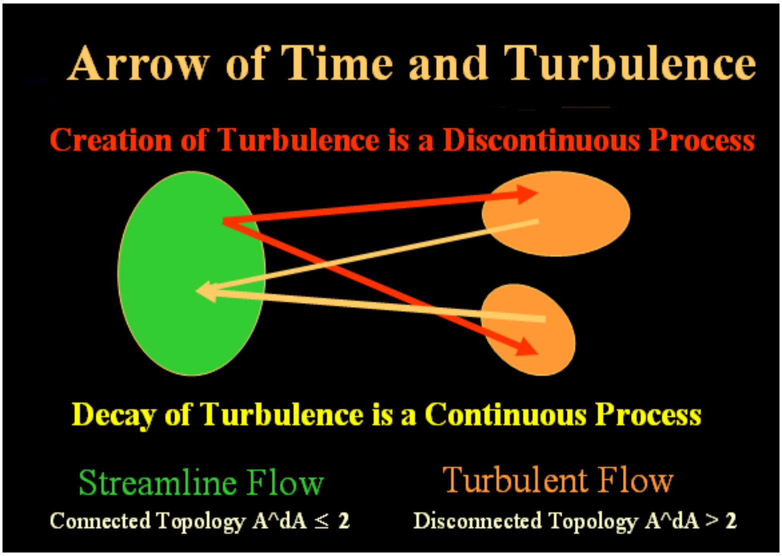
If the homeomorphism is a projection, then $d\phi$ is onto and $d\phi^{-1}$ is 1-1; the covariant rules become uniquely soluble in both a predictive and retrodictive sense.

Finally if the map is a diffeomorphism ($\phi$ exists, $\phi^{-1}$ exists, $d\phi$ is 1-1 and onto, $d\phi^{-1}$ is 1-1 and onto) then both contravariant and covariant rules are uniquely soluble in a predictive and retrodictive sense. The diffeomorphism is the usual map considered in classical tensor analysis as a coordinate transformation. With respect to such maps (that is, with respect to classical tensor analysis) the different solubility features of contravariant and covariant tensor fields becomes degenerate and indistinct.

A summary of the above map classifications is presented in Table I along with the retrodictive or predictive solubility of the associated covariant and contravariant tensor fields. Note the emphasis on retrodiction of covariant fields for irreversible maps.

### 3.2 Irreversibility and Continuous Topological Evolution

On a given domain, Baldwin has shown that the existence of a Cartan 1-form and its Pfaff sequence (4.8) may be used to define a "Cartan" topology over the domain (See Chapter 4 for details). If the Cartan 1-form is integrable in the sense of Frobenius, then the Cartan topology is a connected topology. If the Cartan 1-form is not integrable then the Cartan topology is a disconnected topology.




Evolution from a connected topology to a disconnected topology can proceed only by means of discontinuous processes. However, evolution from a disconnected topology to a connected topology can be accomplished continuously. It is this latter class of continuous processes that will be in focus in this monograph. An important practical result of this fact is that the continuous hydrodynamic evolution from a streamline (integrable and reversible) state to a turbulent ( non integrable and irreversible) state is impossible, while the continuous evolution from a turbulent state to a streamline state is permissible [201]. The creation of irreversible turbulence is necessarily discontinuous, but the decay of turbulence can be continuous.

It has been demonstrated ([172]) that for continuous but non homeomorphic maps (C1 maps without continuous inverse) it is **impossible to predict** the functional form of either covariant or contravariant vector fields. That is, the functional form of tensor fields on the final state is not well defined in terms of the functional form of the field on the initial state, if the map from initial to final state is continuous but not homeomorphic. On the other hand it can be shown that covariant antisymmetric tensor fields are deterministic in a retrodictive sense, even though the continuous maps from initial to final state are not invertible. That is, the functional form of the components of differential forms defined on the final state are well defined on the initial state even if the map from the initial state to the final state is C1 smooth, but not a homeomorphism. With respect to continuous topological evolution there then exists a natural, logical, arrow of time, which is not observable with respect to diffeomorphic geometric evolution that preserves topology. Therefore, to understand irreversible phenomena, a retrodictive point of view seems to be of some value [235] and it is this non statistical retrodictive point of view constructed on Cartan's theory of exterior differential systems that is the point of departure in this section.

The methods will be restricted at first to those evolutionary processes and systems which are C2 continuous. It is appreciated that this restriction does not cover all physical situations, where in the opinion of this author "true" discontinuities, not just mathematical artifacts, are possible. The continuous evolutionary processes to be considered will permit topology to change in a continuous but irreversible manner (example: the pasting together of two blobs, or the collapse of a hole). Discontinuous processes are, at present, excluded.

In this Chapter, a topological perspective will be used to establish the long sought for, non-statistical, connection between dynamic mechanical systems, thermodynamic irreversibility, and the arrow of time. Consider the following remarks concerning the Boltzmann paradox:

> "*The traditional reductionist view is that we should seek the explanation on the basis of the reversible mechanical equations of*

*motion. But, as the physicist Ludwig Boltzmann discovered, it is not possible to base the arrow of time directly on equations that ignore it. His failed attempt to reconcile microscopic mechanics with the second law gave rise to the "irreversibility paradox" Peter Coveny and Roger Highfield, "Chaos, entropy and the arrow of time", (WH Allen, London, 1990; Ballantine, New York, 1991).*

*"The arrow of time is one of the big unclaimed prizes of modern physics. The problem is to reconcile the temporal asymmetry of thermodynamics with the apparent temporal symmetry of fundamental physical theories" Hugh Price Nature 348 (22 November, 1990), p. 356*

On the other hand: *"...when they are correctly presented, the classical views of Boltzmann perfectly account for macroscopic irreversibility on the basis of deterministic, reversible, microscopic laws" J Bricmont 1996.*

*Since in the differential equations of mechanics themselves there is absolutely nothing analogous to the Second Law of thermodynamics the latter can be mechanically represented only by means of assumptions regarding initial conditions. L. Boltzmann ([11], p.170)*

*Indeed, the laws of physics are always of the form: given some initial conditions, here is the result after some time.*

*And Fred Hoyle wrote: "The thermodynamic arrow of time does not come from the physical system itself. . . it comes from the connection of the system with the outside world"*

In this section, the Boltzmann paradox will be resolved in terms of the point of view of Continuous Topological Evolution (See Vol. 1). First consider the definitions:

1. Causal evolution is defined as a map of C1 functions from a domain of base variables to a unique range of base variables. The maps may be many to one and are not necessarily homeomorphisms.

2. Prediction implies that well behaved functional forms (not just numeric point data) on the range of base variables can be deduced from functional forms defined on the domain of base variables.

3. Retrodiction implies that functional forms on the domain can be deduced from functional forms on the range.

The fundamental axioms are:

**Axiom 1**. *Thermodynamic physical systems can be encoded in terms of a 1-form of covariant Action Potentials,* $A_k(x, y, z, t...)$*, on a* $\geq 4$ *dimensional abstract variety of ordered independent variables,* $\{x, y, z, t...\}$*. The variety supports a differential volume element* $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt...$

**Axiom 2.** *Thermodynamic processes are assumed to be encoded, to within a factor,* $\rho(x, y, z, t...)$*, in terms of contravariant vector direction fields,* $V_4(x, y, z, t...)$*.*

**Axiom 3.** *Continuous topological evolution of the thermodynamic system can be encoded in terms of Cartan's magic formula (see p. 122 in [133]). The Lie differential, when applied to a exterior differential 1-form of Action,* $A = A_k dx^k$*, is equivalent abstractly to the first law of thermodynamics.*

$$\text{Cartan's Magic Formula } L_{(\rho\mathbf{V}_4)}A = i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A) \tag{3.3}$$

$$\text{First Law} : W + dU = Q, \tag{3.4}$$

$$\text{Inexact Heat 1-form } Q = W + dU = L_{(\rho\mathbf{V}_4)}A \tag{3.5}$$

$$\text{Inexact Work 1-form } W = i(\rho\mathbf{V}_4)dA, \tag{3.6}$$

$$\text{Internal Energy } U = i(\rho\mathbf{V}_4)A. \tag{3.7}$$

**Axiom 4.** *Equivalence classes of systems and continuous processes can be defined in terms of the Pfaff topological dimension.*

The result of employing these axioms will be to demonstrate that:

1. Topological evolution is a necessary condition for thermodynamic irreversibility.

2. Thermodynamic irreversibility is an artifact of topological dimension 4 or more, while topological dimension 3 is a necessary condition for chaos.

3. The assumption of Uniqueness of predicted solutions (which implies a Topological dimension 2 or less) and Homeomorphic evolution are different constraints on classical mechanics that eliminates any time asymmetry.

   The functional forms of tensor fields with arguments as the base variables of the final state are not well defined in terms of the functional forms of tensor fields with arguments as the base variables of the initial state, unless the map from initial to final state is a diffeomorphism, which preserves topology. On the other hand, the functional forms of those tensor fields which are coefficients of exterior differential forms, and with arguments as the base variables of the initial state, are well defined in terms of the functional forms of tensor fields with arguments as the base variables of the final state, even when the map from initial to final state represents topological evolution. Hence an Arrow of Time asymmetry is a logical result when topological evolution is admitted, but does appear if the evolution is restricted to be homeomorphic.

4. The insistence that a unique outcome can be predicted from given initial data implies that the minimum topological dimension for a given geometrical dimension is 2 or less. If the topological dimension is 3 or greater, and if solutions to a particular evolutionary problem exist, then the solution is not unique. Envelope solutions are classic examples of solution non-uniqueness.

The combined thermodynamic-topological perspective presented herein uses the mathematical tools of exterior differential forms to describe the topological features of physical systems, and vector fields to describe the continuous evolutionary processes that may or may not change the topology of the physical system. Examples will demonstrate that topological change is a necessary condition for thermodynamic irreversibility, and continuous topological change establishes the arrow of time.

## 3.3 Topological Tools

### *3.3.1 Topological Structure*

The idea that the presence of a physical system establishes a *topological structure* on a base space of independent variables is different from, but similar to, the geometric perspective of general relativity, whereby the presence of a physical system is presumed to establish a *metric* on a base space of independent variables. The topological features of the physical system are presumed to be encoded in terms of exterior differential forms, which - unlike tensors - are functionally well behaved with respect to differentiable maps that are not invertible. Note that a given base may support many different topological structures; hence a given base may support many different physical systems.

For (smooth) maps, between base sets, that are C1 differentiable, but are not invertible, it is impossible to predict uniquely the functional forms of covariant or contravariant vector fields, constructed over a final base set, in terms of functional forms given on an initial base set [172]. Point-wise values of the tensor fields in certain cases may be predicted, but the functional forms are never predictable with respect to such non-invertible maps. Hence, classical theories based on tensor fields, which can describe geometrical evolution, will fail to describe topological evolution. It may be surprising to note that (with respect to such non-invertible, non-homeomorphic, maps) it is possible to retrodict the functional forms of covariant vectors and contravariant vector densities on the initial base set in terms of the given functional forms on the final base set. For differentiable evolutionary processes that are diffeomorphisms, topology does not change and both prediction and retrodiction of tensor fields is possible. For differentiable evolutionary processes which are not homeomorphisms, topology changes, and deterministic prediction fails, but retrodiction remains possible. Hence the feature of topological evolution imposes a sense of asymmetry with respect to an evolutionary parameter.

### *3.3.2 Continuity*

Although C1 non-invertible maps are not homeomorphisms, and therefore the topology of the initial state and the topology of the final state are not the same, such maps can be continuous. Continuous topological evolution is not an oxymoron, for topological continuity is defined such that the limit points of every subset in the domain (relative to the topology on the initial state) permute into the closure of the subsets in the range (relative to the topology on the final state). The initial and final state topologies need not be the same.

Pasting together is a continuous process for which the topology of the final system state is not necessarily the same as the topology of the initial system state. Separation or cutting into parts is a discontinuous process for which the system topology of the final state is not the same as the system topology of the initial state. The obvious topological property that changes is the number of parts. Projections from higher dimensions to lower dimensions are classic examples of many to one differentiable maps that are not invertible. The obvious topological property that changes is the property of dimension. Consider a flat putty disc in the shape of an annulus. Deform the putty continuously such that the points that make up the central hole are pasted together. On the other hand make an interior cut in a disk of putty and discontinuously separate the points to make a hole. The obvious topological property that changes is the number of holes. (Discontinuous processes are ignored in this presentation.)

### *3.3.3 Cartan's Magic Formula*

Cartan's "magic formula" (a descriptive phrase introduced by Marsden [133]) representing the "evolution" of the 1-form of Action, $A$, with respect to the "flow" generated by the vector field, $V$, is the cornerstone of the development. The Cartan formula does not depend upon connection or metric, and has been called the "homotopy formula" by Arnold [6]. If the coefficient functions of $A$ and $V$ are C2 differentiable then it is possible to prove that Cartan's formula describes continuous evolution. The C2 constraints can be relaxed, but will be studied elsewhere.

Herein, the following definitions are made:

1. The term $W = i(\mathbf{V})dA$ is defined as the inexact 1-form of "virtual work".

2. The function $U = i(V)A$ is defined as the "internal energy".

3. The sum of the two terms, $W + dU$, define the inexact 1-form of "heat", $Q$.

From these definitions, it is apparent that Cartan's magic formula not only represents an evolutionary process, where the process $V$ acts on the physical system $A$ to produce the 1-form of heat, $Q$, but also is formally equivalent to the cohomological description of the First Law of Thermodynamics.

$$\begin{aligned} L_{(\mathbf{V})}A &= i(\mathbf{V})dA + d(i(\mathbf{V})A) = Q \\ &= W + dU = Q \end{aligned} \tag{3.8}$$

In this monograph this formal correspondence is taken seriously. Moreover the formula can be used to determine equivalence classes of evolutionary processes, for a given physical system, which are thermodynamically reversible or irreversible.

The fundamental theme is to study processes that describe continuous topological evolution. Such evolutionary processes are not necessarily invertible and do not admit unique deterministic prediction of tensor fields from initial data. However, they do permit the deterministic retrodiction of tensor fields by means of functional substitution and pullbacks [64]. The magic in Cartan's formula is that it can be used to describe such evolutionary processes where the topology of the initial state is not the same as the topology of the final state, as well as for adiabatic processes for which the topology does not change.

Both the heat and the work 1-forms as defined above are not necessarily exact, and therefore can lead to non-zero cyclic integrals. The symbol $L_{(\mathbf{V})}$ stands for the "Lie derivative" with respect to $\mathbf{V}$,a term evidently coined by Slebodzinsky [222]. The symbol $dA$ stands for the "exterior derivative" of $A$, and the symbol $i(\mathbf{V})A$ is used to designate the "interior product" of the contravariant $\mathbf{V}$ with the covariant $A$ in a tensorial sense, producing a diffeomorphic invariant. However, no constraints of metric or connection are applied a priori to the domain of definition. For more detail see [123] [124]

For physical systems of measurement it is presumed that the ultimate or fundamental domain (or base) of independent variables will be designated by the ordered quadruplet $\{x, y, z, t\}$. Most useful applications will be constructed from both covariant and contravariant vector fields and functions ultimately defined over this base. However, an initial domain of definition may be conveniently of higher dimension; that is, the initial variety may consist of 2n+1 or 2n+2 independent variables. Note that the initial variety may consist of both "coordinates" and "parameters", and the notation is suitable for application of Fiber bundle theory.

### *3.3.4 Thermodynamic Irreversibility* $Q\hat{}dQ \neq 0$

Following the lead of thermodynamic experience, a thermodynamic process which is reversible it to be associated with a heat 1-form, $Q$, which admits an integrating factor. The integrating factor (in thermodynamics) defines the concept of temperature. Therefore, if the heat 1-form does not admit an integrating factor, the thermodynamic process is irreversible [142]. From a topological point of view, the heat 1-form admits an integrating factor if and only if $Q$ satisfies the conditions of the Frobenius integrability theorem, $Q\hat{}dQ = 0$. This definition of thermodynamic irreversibility, when combined with Cartan's magic formula, permits the link to be made between thermodynamics and mechanical systems.

To repeat: It is subsumed that the physical system can be represented by a 1-form of Action, $A$, and a physical process can be represented by the vector field $\mathbf{V}$. As the system ($A$) is propagated via the action of the Lie derivative with respect to the process, $\mathbf{V}$, the outcome is to produce the heat 1-form, $Q$. Hence a simple test for thermodynamic irreversibility of a process acting on a system is given by the equations:

$$Q\hat{}dQ = (L_{(\mathbf{V})}A)\hat{}(L_{(\mathbf{V})}dA) = 0 \supset the\, process\; is\, reversible. \tag{3.9}$$

$$Q\hat{}dQ = (L_{(\mathbf{V})}A)\hat{}(L_{(\mathbf{V})}dA) \neq 0 \supset the\, process\; is\, irreversible. \tag{3.10}$$

The technique is as follows: First start with a reasonable description of a physical system in terms of a 1-form of Action, $A$, and then for a given vector field, $\mathbf{V}$, representing a process, construct $Q$ from Cartan's formula. Finally, use the Frobenius test to see if the given process is reversible or not.

### *3.3.5 The Pfaff Topological Dimension, n, versus Geometrical Dimension, m:*

Of key importance for any particular physical system is the choice of the "correct" 1-form of Action, $A$, which encodes the topological features of a specific physical system. Experience (guesswork) and the degree of agreement with measurement will satisfy the working scientist. By measurement, it is meant that certain geometrical and topological features will be "observable" evolutionary invariants of a process, or of an equivalence class of processes. In physics, the equivalence class of processes is often specified as solutions to a system of partial differential equations; herein, the alternative view is taken that the equivalence class is generated by an exterior differential system of constraints acting on exterior differential forms. Exterior differential systems are, in effect, specifications of topological constraints on the physical system. For example, the constraint $F - dA = 0$ is topologically a constraint that says the 2-form object is globally closed in an exterior differential sense. The 2-form $F$ is constrained to be equal to the limit sets of $A$, relative to the Cartan topology generated by the topological structure of the given 1-form of Action, $A$.

Perhaps one of the most important, and yet easily computed, ideas is the concept of Pfaff Topological dimension, or class, of an exterior differential 1-form, $A$. Recall that dimension is a topological property, hence if the Pfaff Topological dimension changes during an evolutionary process, the process is describing a process of continuous topological change - a necessary requirement for thermodynamic irreversibility. Examples of such processes will be given below. It is also important to remember that the same set of elements can support more than one topology. In the case at hand, the Pfaff Topological dimension of the three 1-forms, $A$, $W$, and $Q$ will be of importance to the analysis of non-equilibrium thermodynamics, and the arrow of time. A given differential form, $A$, geometrically defined in terms of $2n$ differentials and functions on a variety of geometric base dimension $n$, may require $m \preceq 2n$ independent functions and differentials for its topological description. The Pfaff topological dimension can be less than the Geometric dimension of the space over which the 1-form of Action has been constructed. The concept implies that their exists a differentiable projective map from a space of dimension $2n$ to the space of dimension $m$. An exterior differential form defined on the (final) target space induces a functionally well defined exterior differential form on the (initial) domain space, by means of functional substitution and the "pullback" of the projection. The topological features of investigations on the lower dimensional space can be retrodicted back to the initial higher dimensional manifold, even though the projective mapping is not a homeomorphism and therefor implies topological change. Remarkably, the Pfaff topological dimension of the 1-form on the target space is the same as the Pfaff topological dimension of the pulled back 1-form on the initial space.

For any given 1-form, $A$, functionally defined on a (perhaps geometrical) base space, or variety, of dimension $n$, it is possible to compute the "Pfaff sequence", $\{A, dA, A\hat{}dA, dA\hat{}dA...\}$. It is remarkable that this sequence terminates at a minimum number, $m \leq n$, representing the irreducible minimum number of independent functions required to define the topological encoding. This number $m$ is called the Pfaff topological dimension, and the last non-zero element of the Pfaff sequence is defined as the top Pfaffian. The topological dimension, m, is less than or equal to the geometric dimension, $n$. Note that the requirement for thermodynamic irreversibility implies that the Pfaff dimension of the heat 1-form Q must be 3 or greater, which means the Frobenius theorem of *unique* integrability for the Pfaffian expression, $A = 0$, fails. The idea is that the Pfaff topological dimension implies the existence of a continuous differentiable map from the variety of dimension $n > m$ to a variety of dimension $m$.

An explicit physical 1-form, $A$, will generate a "Cartan topology" on the domain. It is easy to demonstrate that the Cartan topology is a connected topology if the Pfaff dimension is 2 or less, and a disconnected topology if the Pfaff dimension is 3 or more (See Vol 1). Hamiltonian mechanics and Eulerian streamline flows in hydrodynamics (on base spaces of geometric dimension 4) are associated with Action 1-forms of Pfaff topological dimension 2 or less, and are thermodynamically reversible. Turbulence, being the antithesis of streamline flow, must be represented by a topology of Pfaff dimension 4 or more, and is thermodynamically irreversible. Again, the constraint of continuous topological evolution induces a logical "Arrow of Time" related to a change in Topology. Note that the decay of turbulence can be studied by continuous methods, but the creation of turbulence cannot. It is possible to demonstrate that a map from a continuous topology to a discontinuous topology is not continuous, but a map from a continuous topology to a discontinuous topology is continuous. In both evolutionary maps, topological evolution takes place as the Pfaff topological dimension changes, but note that the creation of turbulence cannot be continuous, where the decay of turbulence can be continuous.

When the 1-form A is of odd topological dimension (n= 3 or greater), then the 2-form dA can be put into correspondence with an odd-dimensional $n = 2k+1$ antisymmetric matrix of functions of maximal rank. This matrix has one unique eigenvector with a null eigen value. Hence the topological encoding of a physical system determines a unique direction field defined as the "extremal" direction field (on the $2k+1$ dimensional variety). Evolution in the direction of this unique "extremal" vector field, $\mathbf{V}_E$, implies that the virtual work, W, vanishes, and that the exterior derivative of heat 1-form $dQ$ vanishes, as $Q$ is exact. Such extremal vector fields always have a Hamiltonian generator, and are not thermodynamically irreversible, as $Q\hat{}dQ = 0$. The extremal Hamiltonian evolution preserves the even dimensional topological features of the physical system (the Poincare invariants). If the extremal process is also adiabatic, such that both $Q = 0$ and $dQ = 0$, then the process preserves both odd and even topological features, and is a homeomorphism. The equivalence class of processes that satisfy the closure requirement, ($dQ = 0$) includes not only extremal fields, $i(\mathbf{V}_E)dA = 0$, but also those that can have a Casimir generator (Bernoulli flows) or those that can generate limit cycles.

### *3.3.6 Symplectic manifolds:*

When the 1-form A is of even topological dimension (n = 4 or greater), then the 2-form dA can be put into correspondence with an even-dimensional antisymmetric matrix of functions of maximal rank. Extremal fields (with null eigenvalue) do not exist, but there is a unique evolutionary direction field, $\mathbf{V}_T$, on the $n = 2k + 2$ dimensional variety that is completely determined from the topology of the physical system, induced by the 1-form, A. On a n = 4 dimensional base manifold, this unique direction field is defined by the equations,

$$A\hat{}dA = i(\mathbf{V}_T)dx\hat{}dy\hat{}dz\hat{}dt. \tag{3.11}$$

This vector field $\mathbf{V}_T$ is defined as the Topological Torsion vector on the 2n+2 dimensional Symplectic manifold. As $A\hat{}A\hat{}dA = 0$ the Topological Torsion vector is transverse with respect to the 1-form of Action: $i(\mathbf{V}_T)A = 0$. By direct calculation it is possible to show that $W = i(\mathbf{V}_T)dA = \Gamma\, A$. In other words the 1-form of virtual work is proportional to the 1-form of Action. Cartan's magic formula becomes

$$L_{(\mathbf{V}_T)}A = \Gamma\, A, \tag{3.12}$$
$$\text{with } i(\mathbf{V}_T)A = 0. \tag{3.13}$$

Here $\Gamma$ equals 1/2 the coefficient of the non-zero 4-form

$$dA\hat{}dA = 2 \cdot \Gamma(x, y, z, t) dx\hat{}dy\hat{}dz\hat{}dt \tag{3.14}$$
$$= \{div_4(\mathbf{V}_T)\}dx\hat{}dy\hat{}dz\hat{}dt. \tag{3.15}$$

As the 2-form is of maximal rank, $\Gamma(x, y, z, t) \neq 0$. It follows that evolution in the direction of the Torsion Vector $\mathbf{V}_T$ is thermodynamically irreversible, as

$$Q\hat{}dQ = L_{(\mathbf{V}_T)}A\hat{}L_{(\mathbf{V}_T)}dA = \Gamma^2\, A\hat{}dA \neq 0. \tag{3.16}$$

The factor $\Gamma^2$ plays a role related to the entropy production rate.

If the evolutionary process proceeds to a domain where $\Gamma(x, y, z, t) \Rightarrow 0$, then (on the 2n+2 =4 dimensional Symplectic space) the Topological Torsion vector $\mathbf{V}_T$ satisfies the equations

$$L_{(\mathbf{V}_T)}A = 0, \tag{3.17}$$
$$\text{with } i(\mathbf{V}_T)A = 0, \tag{3.18}$$
$$\text{and } i(\mathbf{V}_T)dA = 0, \tag{3.19}$$
$$\text{or } \mathbf{V}_T \Rightarrow \mathbf{C}_T. \tag{3.20}$$

The Topological Torsion vector $\mathbf{V}_T$ becomes a characteristic direction field $\mathbf{C}_T$ and evolution in the direction of $\mathbf{C}_T$ is no longer thermodynamically irreversible. Note that in this case $dA\hat{}dA \Rightarrow 0$, hence the Pfaff Topological dimension is reduced from $2n + 2 = 4$ to $2n + 1 = 3$. (The arguments can be extended to arbitrary n).

### *3.3.7 Cartan's development of Hamiltonian systems.*

Recall that Cartan proved that if the 1-form of Action is taken to be of the classic format, $A = p_k dq^k + H(p_k, q^k, t)dt$, on a 2n+1 dimensional domain of variables $\{p_k, q^k, t\}$, then a subset of all vector fields, $V$, that satisfy his magical equation would generate "Hamiltonian flows" of classical mechanics.[38] The necessary and sufficient constraint for the vector field to be of the Hamiltonian format was that the closed integrals of the Action $\int_{z1} A$ must be evolutionary invariants of the process, $V$.

$$\text{Cartan's Constraint:} \qquad L_{(\mathbf{V})} \int_{z1} A \Rightarrow 0. \tag{3.21}$$

The symbol, $\int_{z1}$, is used to designate that the integration chain is a closed cycle, z1; $\int_{z2}$, would be used to designate a two dimensional closed cycle; etc.. The cycle may or may not be a boundary.

The Cartan criteria does not constrain the Hamiltonian function $H(p_k, q^k, t)$ to be independent from time, but as will be described below, it does insure that the Cartan topology of the initial state is the same as the Cartan topology of the final state. The same criteria to generate "Hamiltonian flows" can be used on 2n+2 dimensional domains, $(p_k, q^k, t, s)$. The key difference is that on the odd dimensional domain ( a contact manifold) the Hamiltonian flow is a unique "extremal" field. The generator of the flow is the Hamiltonian function, $H(p_k, q^k, t)$. On the 2n+2 dimensional domain (a symplectic manifold), a unique extremal field does not exist. There do exist (many) "Hamiltonian flows", but they are generated, not from $H(p_k, q^k, t, s)$, but from other functions, known as Bernouilli-Casimir functions, $\Theta$. There does, however, exist a unique vector direction field of evolution on the symplectic 2n+2 domain, but it is not a Hamiltonian flow. In fact, it will be demonstrated below that this unique vector field (defined as the Topological Torsion current) represents thermodynamically irreversible processes.

### *3.3.8 Thermodynamic Reversibility* $Q\hat{}dQ = 0$

The Cartan constraint $(L_{(\mathbf{V})} \int_{z1} A = 0)$ thereby partitions all possible vector fields of evolution into two equivalence classes, those representing processes that were "Hamiltonian", and those that were not Hamiltonian. Herein the idea is to exploit the Cartan's magic formula to obtain a better understanding of the non-Hamiltonian processes $(L_{(\mathbf{V})} \int_{z1} A \neq 0)$, and how they may represent dissipative and irreversible physical phenomena. Hamiltonian processes may be time-dependent, hence decaying energy alone is not a sufficient criteria to insure thermodynamic irreversibility.

Following the lead of thermodynamic experience, a thermodynamic process which is reversible it to be associated with a heat 1-form, $Q$, which admits an integrating factor. The integrating factor (in thermodynamics) defines the concept of temperature. Therefore, if the heat 1-form does not admit an integrating factor, the thermodynamic process is irreversible. From a topological point of view, the heat 1-form admits an integrating factor if and only if $Q$ satisfies the conditions of the Frobenius integrability theorem, $Q\hat{}dQ = 0$. This definition of thermodynamic irreversibility, when combined with Cartan's magic formula, permits the link to be made between thermodynamics and mechanical systems.

Rather than applying the method to many examples, it is possible to consider equivalence classes determined by the Pfaff dimension, or class, $Pfaff(W)$, of the 1-form of virtual work, $W$. The Pfaff dimension of the virtual work 1-form, $W = i(\mathbf{V})dA$, depends on both the process ($\mathbf{V}$) and the system ($A$). Conservative Hamiltonian processes belong to the $Pfaff(W) = 0$ or $Pfaff(W) = 1$. Processes that belong to the $Pfaff(W) = 4$ are always irreversible.

## 3.4 The Pfaff Dimension of the 1-form of Virtual Work, Pfaff(W)

Given any 1-form, $W = i(\mathbf{V})dA$, defined over a set of independent variables, it is always possible to construct its Pfaff sequence from the form, $W$, its exterior differential, $dW$, and algebraic exterior products of these objects;

$$\text{Pfaff sequence of W} \quad \{W, dW, W\hat{}dW, dW\hat{}dW, ..\}. \tag{3.22}$$

At some integer M+1 the remaining elements of the Pfaff sequence are zero. The Pfaff dimension, or class, $Pfaff(W)$ of the form, $W$, is defined as the integer M equal to the number of non-zero terms in the Pfaff sequence. This integer is always less than or equal to the number of independent variables. The Pfaff dimension specifies the irreducible number of functions required to specify the 1-form of interest. As the 1-form of Work is constructed from the 1-form of Action, the number of contravariant components of a vector field, $V$, required to define the 1-form of virtual work, $W$, need not exceed the Pfaff dimension of the Action 1-form. However, the components of an arbitrary contravariant vector field on the original domain of definition may not be fully expressible in terms of projected functions of the 1-form of Action. In modern language, the Pfaff dimension of the 1-form of Action determines the base, but the contravariant vector field has additional components along the fibers of the vector bundle.

Cartan's magic formula takes note of this difference, for the 1-form of virtual work, $W$, is transversal to the process, while the 1-form of heat is not.

$$i(\mathbf{V})W = i(\mathbf{V})i(\mathbf{V})dA = 0 \quad but \quad i(\mathbf{V})Q = i(\mathbf{V})d(i(\mathbf{V})A) \neq 0 \tag{3.23}$$

In the language of fiber bundles, this result gives a precise definition to the differences between the concepts of work and heat. Heat can have components along the fibers, work does not.

In that which follows, the features of the various equivalence classes defined by the Pfaff dimension of the 1-form of virtual work are explored. In all classes considered, the trivial case $dA = 0$, is ignored, for then every vector field representing a process on such physical systems is such that the virtual work vanishes. All such cyclic processes are adiabatic, and if the process is such the the internal energy is constant, $dU = d(i(\mathbf{V})A) = 0$, then such processes are locally and globally adiabatic. If the process is an associated vector, (such the $U = i(V)A \Rightarrow 0$) then the process resides on the "equipotential" surface defined by the Pfaffian equation, $dA = 0$.

### *3.4.1 Reversible Case 1: Pfaff(W) = 0. Cyclic processes that are adiabatic and extremal.*

When $dA \neq 0$, the constraint, $Pfaff(W) = 0$, implies that the virtual work 1-form vanishes, $W = i(\mathbf{V})dA = 0$, and the 2-form $dW \Rightarrow 0$. Recall that the 2-form of "vorticity", or field intensities, $dA$, consists of an anti-symmetric matrix of coefficients. Hence, only when the Pfaff dimension of the Action is an odd-integer, 2n+1, is it possible for work 1-form to vanish. In such cases the processes, $\mathbf{V}$, are defined as extremals (a word borrowed from the calculus of variations) and are *uniquely* determined (to within a projective factor) as the null eigen vector of the anti-symmetric matrix of functions that are used to represent the coefficients of the 2-form $dA$. As this extremal constraint determines the "equations of motion", it should be noted that there is a large equivalence class of physical systems that will have the "same" orbital motion. In the extremal case the 1-form of Action is not unique, for any closed 1-form, $\gamma$, with $d\gamma = 0$, may be added to the initial 1-form, $A$, without changing the structure of the 2-form, $dA$. It is the form $dA$ that determines the virtual work, $W$.

$$dA = d(A_0 + \gamma) = dA_0 + d\gamma = dA_0. \tag{3.24}$$

The "equations of motion" are said to be "gauge" invariant in the sense that the virtual work 1-form is the same for all physical systems which are elements of the large equivalence class of Actions which differ from one another by a closed 1-form (the "gauge"). Note that the gauge differences between the elements of different actions are not necessarily exact differentials; the class of actions that produce gauge invariant fields, or equations of motion, can belong to different cohomology classes. In short, the same $W$ has many precursors $A$

However, from a thermodynamic point of view, the heat 1-form, $Q$, and how the system interacts with its surroundings, is sensitive to the closed 1-form additions to the Action 1-form. The heat 1-form, $Q$, and the internal energy, $U$, are **not** gauge invariant.

$$\begin{aligned} L_{(\mathbf{V})}A &= i(\mathbf{V})dA + d(i(\mathbf{V})A) \\ &= 0 + dU = d\{i(\mathbf{V})A_0 + (i(\mathbf{V})\gamma\} = Q \end{aligned} \tag{3.25}$$

However, what is remarkable, is that any closed integral of the Action, $\int_{z1} A$, is a (relative) integral invariant of the extremal evolutionary processes generated by $\mathbf{V}$ of this equivalence class.

$$L_{(\mathbf{V})} \int_{z1} A = \int_{z1} Q = 0 \quad (W = 0). \tag{3.26}$$

Hence, any cyclic integral of the heat 1-form is "gauge" invariant. During portions of the cycle, $Q$ may be positive and negative, such that over the cycle, the net $Q$ is zero. (Such systems are sometimes called "breathers" and can be related to limit cycles that occur in dissipative systems.)

If $Q$ vanishes identically, the process is said to be locally adiabatic. For a given system, the constraint that the process be locally adiabatic, can be satisfied by an extremal vector field, which is also "associated". The two constraints,

$$(i(\mathbf{V})A) = 0 \ \ (associated) \tag{3.27}$$

$$i(\mathbf{V})dA = 0 \ \ (extremal) \tag{3.28}$$

form a subclass of processes defined as "characteristic" processes, It follows that such characteristic processes are locally adiabatic.

As mentioned above, on an even dimensional manifold of maximal rank, extremal fields do not exist. However, as will be discussed below, on the 2n+2 symplectic manifold for which there is no unique extremal field, there does exist a unique direction field defined as the (topological) torsion current. Evolution in the direction of the torsion current can decay to a domain where (through topological evolution) the 2n+2 domain becomes a 2n+1 domain, and from then on the evolution can take place along an extremal direction. The initial decay is essentially a transient process that dies out (irreversibly) to a steady state conservative process.

The extremal evolutionary processes form the basis for classical mechanics on state-space. It is apparent that the net heat around any closed path is cyclically zero. If in addition the internal energy is a constant, $dU = 0$,then such processes are locally adiabatic, $Q = W + dU = 0 + 0 = 0$. As the extremal vector field is determined only up to a factor, $\rho$, it is possible to choose this function such that the internal energy is a constant,

$$U = \rho(i(\mathbf{V})A) = Const. \tag{3.29}$$

For such choices of $\rho$ the extremal process is locally adiabatic.

Suppose the initial domain of independent variables $\{E, t, p_k, q^k\}$ is of dimension 2n+2, with a Darboux representation for the 1-form of Action given by the expression

$$A = p_k dq^k - Edt. \tag{3.30}$$

The top Pfaffian, $dA\hat{}dA...$ is a 2n+2 form

$$dA\hat{}dA... = dE\hat{}dt\hat{}dp_1...\hat{}dp_n\hat{}dq^1...\hat{}dq^n. \tag{3.31}$$

If the Pfaff dimension of the Action 1-form is to be 2n+1, then this 2n+2 form must vanish. Hence the variable, $E$, cannot be functionally independent from the remaining (presumed to be independent) variables; it follows that $E = H\{p, q, t\}$ on the 2n+1 dimensional domain. The Action 1-form is then written in the Cartan-Hamiltonian format

$$A = p_k dq^k - H\{p, q, t\}dt \tag{3.32}$$

Relative to the 2n+1 "coordinates" $\{p_k, q^k, t\}$, consider the vector field $\mathbf{V} = \{f_k, V^k, 1\}$ and find the solution to the equation, $W = i(\mathbf{V})dA = 0$. The result is

$$\mathbf{V} = \{f_k = -\partial H/\partial q, V^k = \partial H/\partial p, 1\} \tag{3.33}$$

and the extremal field is said to be Hamiltonian.

By constructing the exterior derivative of Cartan's magic formula,

$$\begin{aligned} L_{(\mathbf{V})}dA &= di(\mathbf{V})dA + dd(i(\mathbf{V})A) \\ &= dW + 0 = dQ \end{aligned} \tag{3.34}$$

As $dW = 0$ for $Pfaff(W) = 0$, it follows that $dQ = 0$. Hence, all even dimensional elements of the Pfaff sequence generated by the Action, $\{dA, dA\hat{}dA, ...\}$, and their integrals, are absolute invariants of the equivalence class of extremal fields, a result known to Poincare.

Note that all processes for which the work 1-form is of Pfaff class 0 are reversible, for $Q\hat{}dQ = 0$.

### *3.4.2 Reversible Case 2: Pfaff(W) = 1. Symplectic processes.*

When the virtual work 1-form is closed but not zero, $W \neq 0, dW = 0$, then the Pfaff dimension is equal to 1. The closure constraint forces the virtual work 1-form to be composed of a perfect differential and/or a harmonic part. When the virtual work 1-form is exact such that

$$W = i(\mathbf{V})dA = d\Theta(x, y, z, t), \tag{3.35}$$

then the function $\Theta$ is defined as a Bernouilli-Casimir function, and is an invariant (first integral) of those evolutionary process, $\mathbf{V}$, that belong to the $Pfaff(W) = 1$.

$$L_{(\mathbf{V})}\Theta = i(\mathbf{V})d\Theta = i(\mathbf{V})(i(\mathbf{V})dA) = 0. \tag{3.36}$$

In hydrodynamics, the Bernoulli function is a constant along any streamline, but neighboring streamlines will have different values for the Bernoulli function, hence $\Theta = \Theta(x, y, z, t)$ but $i(V)d\Theta = 0$.

When the virtual work 1-form is exact, the processes are not only reversible ($dQ = 0$), but they are also cyclically adiabatic. On the other hand, if the virtual work 1-form is closed, but not exact, then the processes, although reversible, are not cyclically adiabatic.

If the Pfaff class of the Action is even, then there exists a unique vector field, $\mathbf{V}$, that defines a locally reversible adiabatic process. For if $Q = 0$,

$$W = i(\mathbf{V})dA = -dU = -d(i(\mathbf{V})A) \tag{3.37}$$

then

$$L_{(\mathbf{V})}A = W + dU = -dU + dU = Q = 0. \tag{3.38}$$

As an example consider the domain $\{x, y, z, t\}$ and the Action $A = \mathbf{A}\bullet\, d\mathbf{r} - \phi dt$. The adiabatic condition becomes the partial differential system

$$-\partial\mathbf{A}/\partial t - grad\phi + \mathbf{V} \times curl\mathbf{A} = -grad(\mathbf{V} \cdot \mathbf{A} - \phi) \tag{3.39}$$

$$\mathbf{V} \cdot (-\partial \mathbf{A}/\partial t - grad\phi) = \partial(\mathbf{V} \cdot \mathbf{A} - \phi)/\partial t \tag{3.40}$$

However, when the work 1-form in not exact, then the process is not cyclically adiabatic, and there will exist non-zero cyclic contributions to the work and heat. The ratio of these integrals is rational [deRham].

The evolutionary vector field is again said to be "Hamiltonian", for $dp - (-\partial\Theta/\partial q)dt = 0$ and $dq - (\partial\Theta/\partial p)dt = 0$. If the Action is written in the Cartan format,

$$A = pdq - H(p, q, t, \sigma)dt),$$

then the Hamiltonian energy, $H(p, q, t, \sigma)$, is not necessarily an invariant of the flow generated by the Bernoulli-Casimir function, $\Theta$. The Bernoulli-Casimir is, however, an evolutionary invariant, and its gradient is transversal to the evolutionary process.

However, when the work 1-form in not exact, but may have harmonic components, $\gamma$, representing topological obstructions. In these cases, the process is not adiabatic in a cyclic sense, for

$$\int_{z1} Q = \int_{z1} W + d(U) = \int_{z1} \{d(\Theta + U) + \gamma\} = 0 + \int_{z1} \gamma \neq 0$$

There will exist non-zero cyclic contributions to the work and heat. The ratio of these cyclic integrals is rational [deRham].

Note that all processes for which the work 1-form is of class 1 are reversible, for $Q\hat{}dQ = 0$.

### *3.4.3 Case 3: Reversible, Pfaff(W) = 2 or 3*

When $Q\hat{}dQ = 0$, but $dQ = dW \neq 0$, the first law implies that

$$W\hat{}dW + dU\hat{}dW = 0. \tag{3.41}$$

Then either $W\hat{}dW = 0$ (and the Pfaff dimension of $W$ is 2) or $W\hat{}dW \neq 0$ (and the Pfaff dimension of $W$ is 3 or more). In the first case there is a functional relationship between the variables $U = U(P, V)$ or $U = U(T, s)$. In both cases $dW\hat{}dW = 0$, hence the Pfaff dimension of W is 3 or less.

### *3.4.4 Case 4 : Irreversible, Pfaff(W) = 4.*

In this case $dW\hat{}dW = dQ\hat{}dQ \neq 0$ and the process is never reversible. Examine the case where $dA\hat{}dA \neq 0$, on a domain of 4 dimensions. Then there exists a unique direction field $T$ such that

$$A\hat{\ }dA = i(T)dx\hat{\ }dy\hat{\ }dz\hat{\ }dt. \tag{3.42}$$

This vector field $T$ is defined as the Topological Torsion vector. As $A\hat{\ }A\hat{\ }dA = 0$ the Topological Torsion vector is associated with the 1-form of Action:

$$i(T)A = 0. \tag{3.43}$$

By direct calculation it is possible to show that

$$W = i(T)dA = \Gamma\, A. \tag{3.44}$$

In other words the 1-form of virtual work is proportional to the 1-form of Action. Cartan's magic formula becomes

$$L_{(\mathbf{V})}A = \Gamma\, A \tag{3.45}$$

where $\Gamma$ equals the coefficient of the non-zero 4-form

$$dA\hat{\ }dA = \Gamma(x, y, z, t)dx\hat{\ }dy\hat{\ }dz\hat{\ }dt. \tag{3.46}$$

As the 2-form is of maximal rank, $\Gamma(x, y, z, t) \neq 0$.

It follows that evolution in the direction of the Torsion Vector yields

$$Q\hat{\ }dQ = L_{(\mathbf{V})}A\hat{\ }L_{(\mathbf{V})}dA = \Gamma^2\, A\hat{\ }dA \neq 0, \tag{3.47}$$

which implies that the process is thermodynamically irreversible.

## 3.5 Anholonomic Fluctuations.

Consider a physical system that can be defined in terms of the Cartan-Hilbert 1-form of Action,

$$A = L(t; q, v)dt + p_k(dq^k - v^k dt), \tag{3.48}$$

defined on the 3n+1 variety $\{t; q^k, v^k, p_k\}$. Do not assume that $p_k$ is constrained to be a jet; e.g., $p_k \neq \partial L/\partial v^k$. Instead, consider $p_k$ to be a Lagrange multiplier to be determined later. It follows that the exact two form $dA$ satisfies the equations

$$(dA)^{n+1} \neq 0,\ but \quad A\hat{\ }(dA)^{n+1} = 0. \tag{3.49}$$

The actual formula for the top Pfaffian (which is of dimension 2n+2 and not 3n+1) is:

$$(dA)^{n+1} = (n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k) \bullet dv^k\}\hat{}\ \Omega_{2n+1}. \quad (3.50)$$

$$where\ \Omega_{2n+1} = \Omega_p\hat{}\ \Omega_q\hat{}dt, \quad (3.51)$$

$$and\ \Omega_q = dq^1\hat{}...\hat{}dq^n, \quad (3.52)$$

$$and\ \Omega_p = dp_1\hat{}...\hat{}dp_n. \quad (3.53)$$

It is to be noted that the unconstrained top Pfaffian of the Cartan-Hilbert Action is always associated with a symplectic (even dimensional) manifold, but not of the maximum dimension of the space of the 3n+1 variables. For n = 3 degrees of freedom, the top Pfaffian indicates that the topological of Pfaff dimension of the 2-form, $dA$ is $2n + 2 = 8$.

If the domain of definition is constrained such that the momenta are defined canonically, $\partial L/\partial v^k - p_k = 0$, then the 2-form $dA$ is not symplectic on its maximal dimension 2n+2, but becomes a contact structure on 2n+1 with the formula

$$A\hat{}(dA)^n = n!\{p_k v^k - L(t, q, v\}\Omega_{2n+1}. \quad (3.54)$$

The coefficient in brackets is the Legendre transform of the Lagrangian producing the format of the classic Hamiltonian energy. The resulting 2n+1 (state) space always has a contact structure if the "total energy" is never zero, and the momenta are canonically defined. The space is reducible to a 2n phase space only if the Lagrangian is homogeneous of degree 1 in the $v^k$, otherwise it is a contact structure of dimension 2n+1.

Consider evolutionary processes defined in terms of a vector field $\gamma\mathbf{W} = \gamma\{1, v, a,\ f\}$, relative to $\{t; q, v, p\}$. Construct the 1-form $W$ of virtual work by contracting the exact two form $dA$ with the vector field. For every case, the 1-form of virtual work has the format

$$W = i(\mathbf{W})dA = \{p - \partial L/\partial v\}\Delta v + \{f - \partial L/\partial x\}\Delta q. \quad (3.55)$$

where

$$\Delta v = dv - adt \neq 0 \quad (3.56)$$

and

$$\Delta q = dq - vdt \neq 0. \quad (3.57)$$

When the 2-form $dA$ is symplectic, the work 1-form (which can not vanish) has two terms for any n; the first involves $\Delta v$ and the second involves $\Delta q$. The work 1-form cannot vanish if $dA$ is symplectic for there are no null eigenvectors of an anti-symmetric matrix of maximal rank. This fact implies that the following 4 situations are NOT allowed when $dA$ is symplectic.

1. $\{p - \partial L/\partial v\} = 0$ and $\{f - \partial L/\partial x\} = 0$

   (Canonical momentum and gradient forces.)

2. $\{p - \partial L/\partial v\} = 0$ and $\Delta q = 0$

   (Canonical momentum and zero kinematic fluctuations in position.)

3. $\Delta v = 0$ and $\{f - \partial L/\partial x\} = 0$

   (Zero kinematic fluctuations in velocity and gradient forces.)

4. $\Delta v = 0$ and $\Delta q = 0$

   (Zero kinematic fluctuations in velocity and Zero kinematic fluctuations in position.)

Conversely, when $dA$ generates a contact manifold, one of the four cases above must be true. An elementary case is based upon the assumption that 4 is valid. That is, there exists a kinematic description of the process at both the first and the second order. Another case that is common is based on the assumption that the momentum is canonically defined. Then, for the Contact extremal case to exist, and as $p - \partial L/\partial v = 0$, it is necessary that the work 1-form reduces to vanishing expression

$$W = \{f - \partial L/\partial x\}\Delta q \Rightarrow 0 \text{ in the extremal case.} \tag{3.58}$$

The extremal constraint is satisfied when the bracket factor vanishes, which is then the equivalent of the Lagrange-Euler equations of classical mechanics. However, the Contact constraints are also satisfied when the force is a gradient field, or there exist zero fluctuations in position, or the non-zero components of the force (the otherwise dissipative components) are orthogonal to the kinematic fluctuations in position.

Of current interest are those situations when the work one form is closed, but not zero. Such constraints define symplectic (not extremal) evolutionary processes which occur on even dimensional symplectic manifolds. Locally, as $W = i(\mathbf{W})dA = d\Theta$, it can be shown that such evolutionary fields belong to Lie groups, and that the non-constant functions, $\Theta$, are Casimirs.

A hydrodynamicist would use a different set of words. He would describe the Casimir as a Bernoulli function, a function which is constant along a particular flow line, but which will vary from flow line to flow line. Symplectic processes create conservation theorems of the Helmholtz type (conservation of vorticity, conservation of angular momentum..). In such systems, the Hamiltonian energy need not be an evolutionary invariant, and the system can decay to singular points where the symplectic structure condition fails. Such points will be defined as "equilibrium" points of a symplectic process. An example is given in Vol. 1, with more detail given in Vol 3. to show how the Navier-Stokes equations generate evolutionary vector fields of the symplectic type, but the Euler equations (without pressure) generate extremal vector fields.

If $\Delta v$ is interpreted as "anholonomic differential fluctuations" in velocity, and $\Delta q$ is interpreted as "anholonomic differential fluctuations" in position, then it is intuitive to state that fluctuations in velocity relate to temperature and fluctuations in position relate to temperature. Following this train of thought implies that the first term in the expression for $W$ must be related to Enthalpy (functions of the type $-TS$ that involve temperature) and the second term to Helmholtz free energy (functions of the type $+PV$ that involve pressure). The combination defines the Gibbs free energy (functions of the type $-TS+PV$) of closed thermodynamic systems, and reversible processes. These thermodynamic ideas, more than 100 years old, are essentially the Casimirs of the symplectic vector fields of irreducible dimension 2n+2, and are not evident in extremal systems. When the evolutionary vector fields are symplectic, such that $dW = dQ = 0$, they define thermodynamic reversible processes. The Cartan evolutionary equation of a symplectic process becomes

$$\begin{aligned} L_{(\mathbf{W})}A &= W + dU = d\Theta + dU \\ &= \{p - \partial L/\partial v\}\Delta v + \{f - \partial L/\partial x\}\Delta q + dU \qquad (3.59) \\ &\Rightarrow d(-TS + PV + U) = d(G) = Q, \qquad (3.60) \end{aligned}$$

which defines the heat 1-form $Q$ as the "gradient" of the Gibbs free energy, $G = TS - PV + U$. The Gibbs function is an evolutionary invariant by construction, for all Bernoulli-Casimir functions have transversal gradients.

$$L_{(\mathbf{W})}(G - U) = i(\mathbf{W})d(G - U) = i(\mathbf{W})i(\mathbf{W})dA = 0. \qquad (3.61)$$

Under the classic assumption that $dU - TdS + PdV = Q$, it follows that the symplectic evolution generates a Pfaffian form of the type $-SdT + VdP = 0$, which if integrated yields Gibbs version of an equation of state.

When the work 1-form is not closed, then the process can become thermodynamically irreversible. In this case, the evolution is on a symplectic manifold, but the process is not symplectic (as $dW \neq 0$). To test for irreversibility, the usual engineering requirement is that the heat 1-form $Q$ does not admit an integrating factor. Hence, as described above, a given process, $\mathbf{W}$, acting on a physical system, $A$, is irreversible when

$$Q\hat{}dQ = L_{(\mathbf{W})}A\hat{}L_{(\mathbf{W})}dA \neq 0. \tag{3.62}$$

It is remarkable that the symplectic systems of irreducible dimension 2n+2 seem to solve the Boltzmann - Loschmidt-Zermelo paradox of why canonical Hamiltonian mechanics does not seem to be able to describe the decay to an equilibrium state, and why the usual (extremal) methods of Hamiltonian mechanics do not give any insight into the concept of Pressure, Temperature, or the Gibbs free energy. It is extraordinary that answers to these 150 year old paradoxes of physics seem to follow without recourse to statistics if one utilizes Gromov's work on symplectic systems.

The interpretation of the fact that the top Pfaffian is of dimension 2n+2 and not 3n+1 is an open problem. The implication is that there must exist 3n+1-2n+2 = n-1 topological invariants in these systems.

## 3.6 Dissipative Evolution to States Far from Equilibrium

### *3.6.1 Electromagnetic Irreversible Process in the direction of the Topological Torsion vector*

On the four dimensional space-time of independent variables, $(x, y, z, t)$ the Electromagnetic 1-form of Action can be written in the form

$$A=\Sigma_{k=1}^{3} A_k(x, y, z, t)dx^k - \phi(x, y, z, t)dt \tag{3.63}$$

which generates the 2-form

$$dA = \mathbf{B}_z dx\hat{}dy ... + \mathbf{E}_x dx\hat{}dt..... \tag{3.64}$$

The 3-form of Topological Torsion becomes

$$A\hat{}dA = i(\mathbf{V}_T)dx\hat{}dy\hat{}dz\hat{}dt = \mathbf{S}^x dy\hat{}dz\hat{}dt..... - hdx\hat{}dy\hat{}dz, \tag{3.65}$$

such that in engineering language,

$$\mathbf{V}_T = -\{(\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi);\mathbf{A}\bullet\mathbf{B}\} \equiv \{\mathbf{S},h\} \text{ and } \Gamma = (\mathbf{E}\bullet\mathbf{B}). \qquad (3.66)$$

The 4-form of Topological Parity becomes

$$dA\hat{}dA = 2(\mathbf{E}\bullet\mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt = (div\,\mathbf{S} + \partial h/\partial t)dx\hat{}dy\hat{}dz\hat{}dt. \qquad (3.67)$$

From (7) and (8), evolution in the direction of the Topological Torsion vector, $\mathbf{V}_T$, is thermodynamically irreversible.

### *3.6.2 Mechanical Irreversible Process in the direction of the Topological Torsion vector*

Again, the 1-form of Action chosen to represent the Mechanical system will be represented in terms of a Lagrangian term, $Ldt$, augmented by a set of topological fluctuations, or Pfaffian expressions, appended to the Lagrangian term with coefficient functions equivalent to Lagrange multipliers. The result is a natural generalization of the Cartan-Hilbert 1-form of Action (see 3.48).

$$A = L(t;q,v)dt + p_n\Delta^n. \qquad (3.68)$$

The differential 1-forms $\Delta^n(t,q,v)$ represent Pfaffian expressions and the coefficient functions $p_k$ are to be treated as LaGrange multipliers:

$$\Delta^n(t,q,v) = A^n_m dq^m - \phi^k dt. \qquad (3.69)$$

It is apparent that the Pfaff Topological dimension of $A$ is 2n+2. Some of the Lagrange multipliers may be canonical (a constraint), but not all. The 1-form generates a 2n+1 form defined as the extended Topological Torsion vector, $\mathbf{V}_T$. The 2n+2 divergence of $\mathbf{V}_T$ generates the dissipation coefficient, $\Gamma$. Evolution in the direction of $\mathbf{V}_T$ is thermodynamically irreversible. The evolution can proceed until the Pfaff topological dimension is 2n+1, and then it is possible that the evolution proceeds without topological change, having formed a long-lived state far from equilibrium.

In the next section, this procedure is applied to the problem of the sliding bowling ball. The generalize coordinates will be the translation direction $x$, the rotational coordinate, $\theta$, the translational speed, $v$, the rotational speed, $\omega$, and the time, $t$. Three topological constraints be imposed anholonomically by the Pfaffian system:

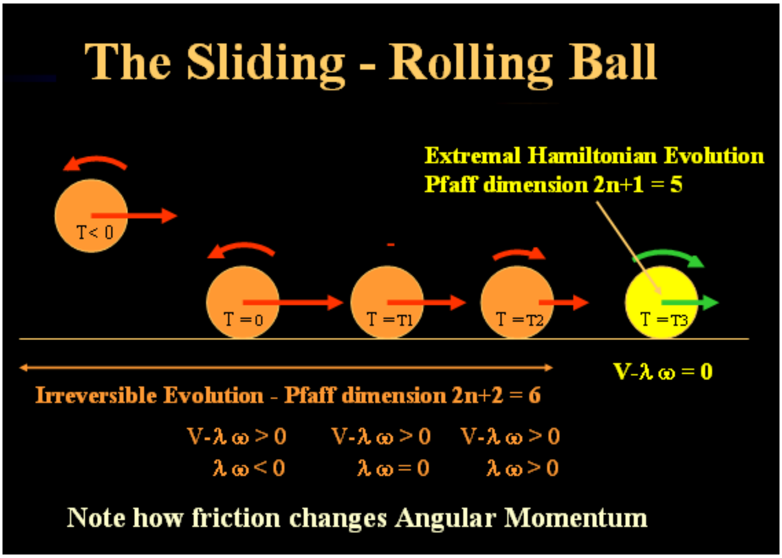
$$\{dx - vdt\} \Rightarrow 0, \quad \{d\theta - \omega dt\} \Rightarrow 0, \quad \{dx - \lambda d\theta\} \Rightarrow 0 \qquad (3.70)$$

## 3.7 An Irreversible Example: The Sliding Bowling Ball

### *3.7.1 The Observation*

Consider a bowling ball given an initial amount of translational energy and rotational energy. Assume the angular momentum and the linear momentum are orthogonal to themselves and also orthogonal to the ambient gravitational field. Then place the bowling ball, subject to these initial conditions, in contact with the bowling alley. Initially, it is observed that the ball slips or skids, dissipating its linear and angular momentum, until the engineering No-Slip condition is achieved. Once the no slip condition is achieved, the ball continues rolling without further sliding and without further irreversible dissipation (ignoring the frictional effect of air viscosity). It is the purpose of this example to demonstrate that the Pfaff Topological dimension of the dissipative irreversible portion of the evolutionary motion is $2n+2 = 6$. When the No-Slip condition is reached, the divergence of the topological torsion tensor on the 6 dimensional space becomes zero. The Pfaff topological dimension of this long-lived No-Slip state is $2n + 1 = 5$, which is far from equilibrium.

Note that it is possible for the angular momentum or the linear momentum to change sign during the irreversible phase of the evolution.



### *3.7.2 The Analysis*

Assume that the physical system may be represented by a 1-form of Action constructed from a Lagrange function:

$$L = L(x, \theta, v, \omega, t) = \{\beta m(\lambda\omega)^2/2 + mv^2/2\} \tag{3.71}$$

The constants are: m=mass, $\beta$ = moment of inertial factor (2/5 for a sphere), $\lambda$ = effective "radius" of the object, the moment of inertia = $\beta m\lambda^2$.

Let the topological constraints be defined anholonomically by the Pfaffian system:

$$\{dx - vdt\} \Rightarrow 0, \quad \{d\theta - \omega dt\} \Rightarrow 0, \quad \{dx - \lambda d\theta\} \Rightarrow 0 \tag{3.72}$$

Define the constrained 1-form of Action as

$$A = L(x, \theta, v, \omega, t)dt + p\{dx - vdt\} + l\{d\theta - \omega dt\} + s\{\lambda d\theta - dx\} \tag{3.73}$$

where $\{p, l, s\}$ are Lagrange multipliers. Rearrange the variables to give (in the language of optimal control theory) a pre-Hamiltonian action:

$$A = (+p - ms)dx + (l + \lambda s)d\theta - \{pv + l\omega - L\}dt, \tag{3.74}$$

$$(\pi_x)dx + (\pi_\theta)d\theta - \{h_t\}dt, . \tag{3.75}$$

It is apparent from the Darboux format that the Pfaff Topological dimension of this Action 1-form is the sum of 3 independent differentials $\{dx, d\theta, dt\}$ and 3 independent functions $\{\pi_x, \pi_\theta, h_t\}$ which is equal to 6.

For simplicity, assume that two of the Lagrange multipliers are defined canonically and are interpreted as canonical momenta; e.g.,

$$p = \partial L/\partial v \Rightarrow mv, \quad l = \partial L/\partial\omega \Rightarrow \beta m\lambda^2\omega \tag{3.76}$$

which implies that

$$A = (mv - s)dx + (\beta m\lambda^2\omega + \lambda s)d\theta - \{mv^2/2 + \beta m(\lambda\omega)^2/2\}dt. \tag{3.77}$$

The volume element of the 6 dimensional symplectic manifold is given by the expression

$$6Vol = 6m^2\beta\lambda^2\{v - \lambda\omega\}dx\hat{}d\theta\hat{}dv\hat{}d\omega\hat{}ds\hat{}dt = dA\hat{}dA\hat{}dA \tag{3.78}$$

The 6D volume element is either expanding or contracting (irreversibly) with a coefficient $6m^3\beta\lambda^2\{v - \lambda\omega\}$. This dissipative coefficient is related to the concept of "bulk" viscosity.

The symplectic manifold has a singular subset upon which the Pfaff dimension of the Action 1-form is 2n+1 = 5. The constraint for such a contact manifold is precisely the No-Slip condition (when the "Bulk viscosity" goes to zero):

$$\{v - \lambda\omega\} \Rightarrow 0. \tag{3.79}$$

This condition is the analogue of the zero divergence condition in incompressible hydrodynamics, only the divergence is that associated with the topological torsion vector, $d(A\hat{}dA\hat{}dA)$ in a six dimensional (2n+2) space.

Motion on the 6 dimensional space cannot be Hamiltonian, for on the 6 dimensional symplectic manifold, there does not exist a unique extremal field, nor a unique stationary field, that can be used to define the dynamics of the physical system. The symplectic manifold does support vector fields, **S**, that leave the Action integral invariant, but these vector fields are not unique in the sense that that they depend on an arbitrary gauge addition to the 1-form of Action that may be required to satisfy initial conditions.

There does exist a unique torsion field (or current) defined (to within a projective factor, $\sigma$) by the 6 components of the 5 form,

$$\text{Topological Torsion} = A\hat{}dA\hat{}dA \tag{3.80}$$

Relative to the topological coordinates $[dx, d\theta, dt, dv, d\omega, ds]$, the Topological Torsion vector has the components

$$\mathbf{T}_6 = [0, 0, 0, \mathbf{T}_v, \mathbf{T}_\omega, \mathbf{T}_s], \tag{3.81}$$

$$A\hat{}dA\hat{}dA = i(\mathbf{T}_6)\Omega_6, \tag{3.82}$$

$$\mathbf{T}_v = m^2\beta\lambda^2\{+\beta\lambda^2\omega^2 + 2\lambda v\omega - v^2\} \tag{3.83}$$

$$\mathbf{T}_\omega = m^2\lambda\{+\beta\lambda^2\omega^2 - 2\beta\lambda v\omega - v^2\} \tag{3.84}$$

$$\mathbf{T}_s = m^2\beta\lambda^2\{+\beta m\lambda^2\omega^2 + mv^2 + 2(\lambda\omega - v)s\} \tag{3.85}$$

If the three non zero components of the Topological Torsion vector are treated as a dynamical system, then it is to be noted that the dynamical system is a Volterra system generated on a Finsler space (see p.205 [4]).

This unique Topological Torsion vector, $\mathbf{T}_6$, independent of gauge additions, has the properties that

$$L_{(\mathbf{T})}A = \Gamma \cdot A \quad and \quad i(\mathbf{T})A = 0. \tag{3.86}$$

This "Torsion" vector field satisfies the equation

$$L_{(\mathbf{T})}A\hat{}L_{(\mathbf{T})}dA = Q\hat{}dQ = (6m^2\beta\lambda^2\{v - \lambda\omega\})^2 A\hat{}dA \neq 0. \tag{3.87}$$

Hence a dynamical system having a component constructed from this unique Torsion vector field becomes a candidate to describe the initial irreversible decay of angular momentum and kinetic energy.

It is to be noted that the non canonical "symplectic momentum" variables, defined by inspection from the constrained 1-form of Action lead to the momentum map:

$$P_x \doteq mv - s, \qquad P_\theta \doteq m\beta\lambda^2\omega + s\lambda. \tag{3.88}$$

Substitution in terms of the momentum variables leads to the generic form (p. 31 [268], also see [132]) for the 1-form of Action:

$$A = P_x dx + P_\theta d\theta - Hdt \tag{3.89}$$

where H is an independent variable on the 6-dimensional manifold. The H map is given by the expression for energy where $v$ and $\omega$ are eliminated in terms of the $P_x$ and the $P_\theta$.

$$H = (mv^2/2 + \beta m(\lambda\omega)^2/2) \Rightarrow (1/2m)[(P_x + s)^2 + (1/\beta)(P_\theta/\lambda - s)^2] \tag{3.90}$$

Note that $v = \partial H/\partial P_x$ and $\omega = \partial H/\partial P_\theta$. Each component of "canonical momenta" decays with the same rate in the canonical domain.

On the 5 dimensional contact submanifold there exists a unique extremal (Hamiltonian) field which (to within a projective factor) defines the conservative reversible part of the evolutionary process. As this unique extremal vector satisfies the equation

$$i(\mathbf{V})dA = 0, \tag{3.91}$$

it is easy to show that dynamical systems defined by such vector fields must be reversible in the thermodynamic sense. (As $dQ = d(i(\mathbf{V})dA) = 0$ for all Hamiltonian or symplectic processes, it follows that $Q\hat{}dQ = 0$.) Note that a particular representation for a Hamiltonian process is given by the Topological Torsion vector $\mathbf{T}_6$ which, in domains of zero divergence, is a characteristic, hence Hamiltonian, vector for the 6 dimensional system constrained to the 5 dimensional hypersurface defined by the no slip condition. This vector is not only Hamiltonian, it is associative, and therefor is representative of an adiabatic Hamiltonian process.

# Chapter 4
# A SUMMARY OF NON EQUILIBRIUM THERMODYNAMICS

This chapter summarizes the concepts of continuous topological evolution as applied to Non Equilibrium Thermodynamics and Processes.

## 4.1 From the Topological Perspective of Continuous Topological Evolution

According to many authors [235] , the connection between deterministic predictive mechanics and thermodynamics remains an open problem. The topological relationships and constraints that make up the laws of equilibrium thermodynamics (ref Tisza [235], Boyling [23]) heretofore have resisted analysis in terms of the geometrical and deterministic methods of classical mechanics. A fundamental issue is that the concept of intensive and extensive variables in thermodynamics is not compatible with Riemannian geometries built on quadratic forms [219]. Although non Riemannian Finsler spaces (projectivized spaces that support torsion and distinguish between functions that are homogeneous of degree 1 and degree 0) appear to be the natural domain for equilibrium thermodynamics, little has been done in this area [45] [4]. Statistical methods appear to lead to reasonable values for equilibrium properties [163] of physical systems, but neither the statistical or deterministic prediction methods of mechanics say anything about the details of the processes - especially of the irreversible processes - that can take place. Other authors have emphasized the topological foundations of thermodynamics [115], and from the time of Caratheodory have noted the connection to Pfaff systems [95]. However, these authors did not have access to, or did not utilize, the Cartan topology and DeRham cohomology. A remark by Tisza, greatly stimulated the early developments of the theory presented in this monograph:

> ”... *the main content of thermostatic phase theory is to derive the topological properties of the sets of singular points in Gibbs phase space*” (p.195 [235] ),

It has been demonstrated [172]) that for continuous but non homeomorphic maps (C1 maps without continuous inverse) it is **impossible** to predict the functional form of either covariant or contravariant vector fields. That is, the functional form of the field on the final state is not well defined in terms of the functional form of the field on the initial state, if the map from initial to final state is continuous but not homeomorphic. On the other hand it can be shown that antisymmetric covariant tensor fields and contravariant tensor densities are deterministic in a retrodictive sense, even though the continuous maps from initial to final state are not reversible. That is, the functional form of the components of differential forms defined on the final state are well defined on the initial state even if the map from the initial state to the final state is C1 smooth, but not a homeomorphism. With respect to continuous topological evolution there exists a natural, logical, arrow of time, which is not observable with respect to diffeomorphic geometric evolution. Therefore, to understand irreversible phenomena, a retrodictive point of view seems to be of some value and it is this non statistical retrodictive point of view constructed on exterior differential systems that is the point of departure in this monograph.

The methods will be restricted at first to those evolutionary processes which are C2 continuous. It is appreciated that this restriction does not cover all physical situations, where in the opinion of this author ”true” discontinuities, not just mathematical artifacts, are possible. The continuous evolutionary processes to be considered will permit topology to change in a continuous but irreversible manner (example: the pasting together of two blobs, or the collapse of a hole). Discontinuous processes are at first excluded.

This first introductory chapter will introduce and describe many of the ideas, and the general philosophy, associated with a topological view of thermodynamics. The reader is to be alerted to the fact that details, derivations and examples are to be found in the subsequent chapters. As implied in the preface, a major objective of this monograph is to establish a topological, non statistical, link between thermodynamics and mechanical, electrical, or hydrodynamic physical systems. A particular goal is to develop a method of describing the differences, and how and when such differences occur, between

- Equilibrium and non equilibrium physical systems, and between

- Reversible and irreversible evolutionary processes acting on such systems.

Warning! The topology of interest is that generated by Cartan's Topological structure, which is defined in terms of his theory of exterior differential forms (See chapter 5). The topology is NOT a metric topology, NOT a Hausdorff topology, and even does NOT satisfy the separation axioms required to be a $T_1$ topology*. Yet all of the pertinent topological ideas, including the non intuitive ones, are easy to grasp from the simple example of the $T4$ point set topology. Of utmost importance, for physical systems of topological dimension greater than 2, the Cartan topology is not a connected topology. As will be discussed below, when the topological dimension is greater than 2, the physical system is a non equilibrium physical system. The importance of this result resides with the topological theorem that mappings from a disconnected topology to a connected topology can be continuous, but continuous maps from a connected topology to a disconnected topology are impossible. In other words *continuous* maps can be used to describe the decay of a Turbulent state, but not its creation [201].

However, the presentation herein is not meant to be a textbook on Cartan's theory of exterior differential forms, nor a textbook on abstract topology. Instead an effort has been made to meld Cartan's methods and topological ideas in a manner that would be useful to the applied researcher and engineer. At the time of writing, not too many physicists, and almost no engineers, are conversant in either the language of the exterior calculus or the language of topology. Of course, some familiarity with the fundamentals of each said discipline is required, and to that end several terse, to the point, presentations (with examples) are given in the Appendix. Most of the useful topological ideas can be rapidly absorbed in terms of point set topology with its metric de-emphasis. In fact, one of the beauties of using the Cartan calculus is that it constructs a differential topology that is free of the metric and connection constraints of differential geometry. It is extraordinary that the Maxwell theory of Electromagnetism (when based on the fields, **E** and **B**, distinct from the fields, **D** and **H**) was one of the first physical theories to be recognized as being a topological theory [245], independent from a choice of metric, or connection. Geometric constraints (such as constitutive constraints between the two distinct sets of fields) merely refine the topological features of the fundamental theory. In this monograph, it will

*For those not familiar with point set topology, chapter 5 in Schaum's outline [125] can be useful.

be demonstrated how thermodynamics may be considered fundamentally as a topological theory, also independent from metric and connection.

Pick up a modern text in classical thermodynamics and note the appearance of the following words used in describing fundamental thermodynamic concepts:

1. Isolated
2. Closed
3. Open
4. Number of disconnected parts (moles)
5. Closed Cycles
6. Integrability
7. Extensive (homogeneous degree 1) variables
8. Intensive (homogeneous degree 0) variables

Now go to Schaum's Outline, "General Topology" [125], or some other textbook on elementary topology, and check the index for these terms. All of these terms have precise definitions in topology, without the imposition of geometric constraints of size, shape or scales. In short, it would appear that Thermodynamics has its foundations in topology, and should be treated as a topological theory from the outset. This is the topological perspective of thermodynamics adopted herein.

In 1974 it was suggested that a certain extension to Hamilton's principle [165], [169] could be made such that the evolutionary processes considered would describe dissipative mechanical systems. Cartan had proved that extremal vector fields, which satisfy the Cartan-Hamilton equation, $i(\mathbf{V})dA = 0$, are generators of Hamiltonian dynamical processes [38]. Rather than study such "extremal" vector fields, it was suggested to consider those processes that satisfy the extended equation: $i(\mathbf{V})dA = \Gamma A + d\theta$. Throughout this current presentation, and in the older articles, it is subsumed that a physical system may be described adequately by a 1-form of Action, $A$, and a physical process may be defined in terms of a dynamical system generated by a vector field, $\mathbf{V}$. It was not appreciated in 1974 that the topological domain of the extremal conservative (Hamiltonian) systems was a contact manifold of odd topological dimension, while the topological domain of the suggested dissipative extension was a symplectic manifold of even topological dimension. (Extremal solutions do not exist on even dimensional manifolds of maximal rank). Currently, the concept of topological dimension seems to be intimately connected to the differences between conservative versus dissipative processes. In fact, as is developed in that which follows, irreversible turbulent thermodynamic processes are artifacts of Pfaff topological dimension 4 (or more). Irreversibility requires that the evolutionary topology of the initial state is not the same as the topology of the final state. It sometimes comes as a surprise to realize that such topological changes can occur continuously. These features of continuous topological evolution (and an explanation of the symbols) are presented in detail in Chapter 5 and 6, below.

A symplectic manifold is defined by the non zero domain of an exact 2-form, $F = dA$. The concept of Hamiltonian mechanics can be extended to symplectic (even dimensional) manifolds, where the Bernoulli-Hamiltonian constraint is of the form $i(\mathbf{V})dA = -dB$, and which also satisfies the more general Helmholtz constraint, $di(\mathbf{V})dA = 0$. Such Helmholtz processes (although reversible) are of interest for they admit the evolution of topological defects of the Bohm-Aharanov type. These topological defects are related to the Work 1-form, $W = i(\mathbf{V})A$, which is produced by the process $\mathbf{V}$ acting on the physical system, $A$. These Work related defects are due to the Work 1-form and are not due to the closed, but not exact, parts of the 1-form of Action.

Almost symplectic manifolds are defined by a 2-form $G$ which is closed but not exact. These even dimensional manifolds $G$ can have compact defect domains without boundary. Such topological structures have been studied by Fomenko [24]. From electromagnetic theory, it becomes apparent that the non exact 2-form (almost symplectic) $G$ is to be associated with the defect structures called charge, and extensive thermodynamic variables, $\mathbf{D}$ and $\mathbf{H}$, while the exact 2-form (symplectic) $F$ is to be associated with intensive thermodynamic variables, $\mathbf{E}$ and $\mathbf{B}$.

The 1-form of Action (Lagrangian) point of view has its advantages, for the fundamental 2-form of a symplectic domain is deduced by construction, $A \Rightarrow F = dA$. The disadvantage is that almost all symplectic domains so constructed are not compact without boundary. This apparent flaw becomes an advantage when it is appreciated that such non compact domains are precisely that which is needed to describe closed (but not isolated) or open thermodynamic systems.

Most classical "laws of physics" are based upon the dogma that a useful physical theory of evolution must give a unique prediction starting from a given set of initial conditions. Combining this constraint with a mathematical description of physical systems in terms of geometrical tensor fields leads to evolutionary processes which preserve topological properties and are (therefore) "reversible". The ubiquitous assumption of uniqueness of predicted solutions, and/or homeomorphic evolution, are topological constraints on "classical mechanics" that eliminates any time asymmetry. The point of departure in this monograph assumes that reasonable physical laws must be capable of describing topological change, and when this feature is encoded in mathematical form, the laws of physics are no longer necessarily reversible. Hence, in this monograph, the Boltzmann paradox will be resolved in terms of a theory based upon *Continuous Topological Evolution* [Chapter 5]. It is presumed that the presence of a physical system establishes a *Topological Structure* [Chapter 4] on a base space of independent (but ordered) variables. When a specific evolutionary process is applied to this physical system, the topology becomes refined.

It will become evident that physical systems require two topological structures, one based upon an exact 2-form, $F = dA$, and its associated symplectic manifold, and the other based upon a non exact 2-form, $G$, which may or may not be closed, but when closed, $dG = 0$, leads to an *almost* symplectic structure with compact topological defects.

## 4.2 Applied topology versus applied geometry

It was mentioned above that the presentation herein is not meant to be a textbook on Cartan's theory of exterior differential forms, nor a textbook on abstract topology. Instead, this monograph is an attempt to use the simpler features of topology contained in the exterior calculus and apply the Cartan methods of continuous topological evolution to interesting non equilibrium problems in the disciplines of mechanics, electrodynamics, or hydrodynamics, without the use of probability theory or statistics. Although the concept of equilibrium has had many useful scientific successes, the truth of the matter is that the observable parts of the real world are rarely in equilibrium. The historical theories and methods of describing evolution that have been developed so far give no details as how the change from non equilibrium to equilibrium takes place. Even the much touted Quantum Mechanics fails to describe the details of the evolutionary decay of an excited state to the ground state. Paraphrasing Bohr, "a miracle takes place" is not a satisfying answer.

The topological methods employed herein can be used to determine when a physical system is in an equilibrium or non equilibrium state. The topological methods employed herein can be used to distinguish thermodynamically irreversible from reversible evolutionary processes. The topological methods employed herein can be used to describe the irreversible dissipative decay processes from open systems into excited stationary states far from equilibrium, and the further decay from excited states into equilibrium ground states.

Since before the beginning of the 20th century, advances in physical theories have been predicated upon a geometrical approach. (It should be mentioned that another interesting point of view about thermodynamics based upon algebra was presented by Zeleznik [266].) Several attempts to better understand thermodynamics in terms of geometrical ideas have been attempted [20], [258], but without notable success. It was pointed out by Tisza [235] that metrical based properties can not be used to distinguish between the two classes of intensive and extensive thermodynamic variables, and the hint was offered that perhaps topological methods, rather than geometrical methods, might prove to be suitable. In fact, as mentioned above, Tisza states the "main content of thermostatic phase theory is to derive the topological properties of the sets of singular points in Gibbs space". These ideas will be examined in detail and extended in sections 2.10 and chapter 3. Finsler spaces have been examined by Anotelli and Ingevarden [4]. At the Aspen conference in 1977, the present author suggested that the methods based upon the first fundamental form (metric) should be replaced by methods based on the second fundamental form (the shape matrix). In section 2.10, these ideas will be exploited in describing how in 4 dimensional space, the van der Waals gas is a universal topological artifact. The same ideas apply to the symplectic dual vector fields that encode dynamical systems. In 4D, the features of the van der Waals gas are universal.

Much of the motivation for development of a topological view of thermodynamics was based upon the concept of topological defects being related to domains or points where topological change took place. The idea that a phase transition was a realization of topological evolution and change was very influential in the struggle over the years to develop a dynamics of such a thermodynamics process. The work of van der Kulk and Schouten inspired the use of the concept of Pfaff topological dimension, and its change, as being one of the key tools for use in topological thermodynamics. The later work Martinet and Zhitomirski [132] [268] has yet to be fully exploited.

The concept that smoothness could influence thermodynamic evolution has only recently been appreciated. How C1 smoothness interplays with the Nash and Gromov axioms has yet to be exploited, but it was a surprise to find out that C1 translational sequences of transitive processes in 3D could be reversible, where C2 intransitive processes of rotation and expansion could be thermodynamically irreversible (see chapter 3)

Herein, the emphasis is on topological properties and features of physical systems, and moreover, how the topology of a physical system can change in a continuous evolutionary manner. What is meant by this statement is that the topology of the initial state need not be the same as the topology of the final state. Such topological change can take place either continuously (pasting) or discontinuously (cutting). The result to be demonstrated is that topological change is a necessary artifact of continuous thermodynamic irreversibility.

## 4.3 Topological Universality

It is a remarkable fact that the physical theories of Thermodynamics, Electrodynamics and Hydrodynamics all have similar topological foundations. These similarity features become evident, and useful, when the different disciplines are expressed in the universal language of Cartan's theory of exterior differential forms.

1. Each discipline utilizes the concept that a physical system can be encoded in terms of an exterior differential 1-form of Action, $A$.

2. Each discipline utilizes the concept that a process, or current, acting on the physical system, can be encoded to within a factor, $\rho$, by a contravariant direction field, $V$.

3. Each discipline has a dynamics that can be expressed in terms of continuous topological evolution based upon the Lie differential with respect to $V$. Warning: this topological dynamics is not always fully equivalent to that dynamics generated by the covariant differential of tensor analysis. The geometric dynamics of tensor analysis is a subset of the topological dynamics.

The arguments of the functions that define the physical system, the process, and the induced additional 1-forms, in this section are limited (with some exceptions) to an ordered variety of $n = 4$ independent base variables, abstractly specified as $\{x, y, z, t\}$, and their differentials, $\{dx, dy, dz, dt\}$. It is presumed that other varieties of base variables $\{\xi^1, \xi^2, \xi^3, \xi^4\}$ can be represented in terms of diffeomorphic maps from $\{\xi^1, \xi^2, \xi^3, \xi^4\}$ to $\{x, y, z, t\}$. To a physicist, the base variables play the role of admissible coordinates if they are diffeomorphically related. However no specific geometric metric or connection is (necessarily) imposed on these varieties of pre-geometric dimension $n = 4$ base variables.

The main thrust of this monograph is to study useful applications of Cartan's theory of exterior differential systems to problems of non equilibrium thermodynamics systems. Some knowledge of Cartan's' topological structure, and the topological properties of the continuum are required. This material has been developed in Chapter 4 and Chapter 5.

## 4.4 Fundamental Axioms and Notable Results

### *4.4.1 Axioms*

**Axiom 1**. *Thermodynamic physical systems can be encoded in terms of a 1-form of covariant Action Potentials,* $A_k(x, y, z, t...)$, *on a* $\geq 4$ *dimensional abstract variety of ordered independent variables,* $\{x, y, z, t...\}$. *The variety supports a differential volume element* $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt...$

**Axiom 2.** *Thermodynamic processes are assumed to be encoded, to within a factor,* $\rho(x, y, z, t...)$, *in terms of contravariant vector direction fields,* $V_4(x, y, z, t...)$.

**Axiom 3.** *Continuous topological evolution of the thermodynamic system can be encoded in terms of Cartan's magic formula (see p. 122 in [133]). The Lie differential, when applied to a exterior differential 1-form of Action,* $A = A_k dx^k$, *is equivalent abstractly to the first law of thermodynamics.*

$$\begin{aligned}
\text{Cartan's Magic Formula } L_{(\rho\mathbf{V}_4)}A &= i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A) && (4.1)\\
\text{First Law } &: W + dU = Q, && (4.2)\\
\text{Inexact Heat 1-form } Q &= W + dU = L_{(\rho\mathbf{V}_4)}A && (4.3)\\
\text{Inexact Work 1-form } W &= i(\rho\mathbf{V}_4)dA, && (4.4)\\
\text{Internal Energy } U &= i(\rho\mathbf{V}_4)A. && (4.5)
\end{aligned}$$

**Axiom 4.** *Equivalence classes of systems and continuous processes can be defined in terms of the Pfaff topological dimension.*

In effect, Cartan's methods can be used to formulate precise mathematical definitions for many thermodynamic concepts in terms of topological properties - without the use of statistics or geometric constraints such as metric or connections. Moreover, the method applies to non equilibrium thermodynamical systems and irreversible processes, again without the use of statistics or metric constraints. The fundamental tool is that of continuous topological evolution, which is distinct from the usual perspective of continuous geometric evolution.

In order to make the equations more suggestive to the reader, the symbolism for the variety of independent variables has been chosen to be of the format $\{x, y, z, t\}$, but be aware that no geometric constraints of metric or connection are imposed upon this variety. For instance, it is NOT assumed that the base variety is euclidean.

It is emphatically stated that geometric notions of scale and metric are to be avoided in favor of topological properties, some of which are invariants of continuous topological evolution, and some of which are not. Those classical thermodynamic features which are diffeomorphic invariants (useful to many equilibrium applications) are ignored, while topological features which are invariants of continuous transformations (and therefore useful to non equilibrium applications) are not. Topological evolution is understood to occur when topological features (not geometrical features of size and shape) change. The motivation for this perspective was based upon the goal of developing analytical methods which could decide if a given physical system was an equilibrium system or a non equilibrium system, and, also, if a specific analytic process was applied to the physical system, was that process reversible or irreversible.

### *4.4.2 Notable Results*

Remarkably, utilization of these (topological) axioms leads to notable results that are not obtained by geometric methods:

1. Thermodynamics is a topological theory.

2. Topological change is a necessary condition for thermodynamic irreversibility.

3. When the Pfaff topological dimension of the 1-form of Action that encodes a physical system is 2, or less, the system is topologically *isolated.* The topological structure on domains of topological dimension $n \leq 2$ never admit a continuous process which is thermodynamically irreversible. non equilibrium systems are of Pfaff topological dimension $> 2$.

4. A 1-form of Action, $A$, with Pfaff topological dimension equal to 1, defines an *equilibrium* isolated system which has representation as a Lagrangian submanifold.

5. The topological structure of physical systems on domains (contact manifolds) of odd topological dimension $n = 3, 5, 7.. > 2$ are non equilibrium systems. On such systems there exists (to within a factor) a unique continuous extremal process, $V_E$, which may be chaotic, but nevertheless is thermodynamically reversible, and has a Hamiltonian generator.

6. The topological structure of physical systems on domains of even topological dimension $n = 4, 6, 8... > 2$ (symplectic manifolds) are non equilibrium systems. Such systems support (to within a factor) a unique continuous process, $V_T$, related to the concept of Topological Torsion. Continuous evolution in the direction of the topological torsion vector is thermodynamically irreversible. In this sense, thermodynamic irreversibility is an artifact of topological dimension $n \geq 4$.

7. The change of the Pfaff topological dimension will produce topological defects and thermodynamic phase changes.

8. The assumption of uniqueness of evolutionary solutions (which implies a Pfaff Topological dimension equal to 2 or less), and homeomorphic evolution, are different, but ubiquitous, constraints imposed upon classical mechanics that eliminate any time asymmetry.

9. All Hamiltonian, Symplectic-Bernoulli and Helmholtz processes are thermodynamically reversible. In particular, the work 1-form, $W$, created by Hamiltonian processes is of Pfaff topological dimension 1 or less.

10. The functional forms of tensor fields with arguments in terms of the base variables of the final state are not deterministically predictable in terms of the functional forms of tensor fields with arguments in terms

of the base variables of the initial state, unless the map from initial to final state is a diffeomorphism (which preserves topology) [172] . On the other hand, the functional forms of those alternating tensor fields which are coefficients of exterior differential forms, and with arguments in terms of the base variables of the initial state, are well defined in terms of the functional forms of tensor fields with arguments in terms of the base variables of the final state, even when the (C1) map from initial to final state describes topological evolution. In other words, retrodiction of differential forms is possible when topology changes, but prediction is impossible. Hence an Arrow of Time asymmetry is a logical result [206] when topological evolution is admitted, but does not appear if the evolution is restricted to be homeomorphic, and therefor topologically invariant.

11. The topological structure of domains of Pfaff dimension 2 or less creates a connected, but not necessarily simply connected topology. Evolutionary solution uniqueness is possible.

12. The topological structure of domains of Pfaff dimension 3 or more creates a disconnected topology of multiple components. If solutions to a particular evolutionary problem exist, then the solutions are not unique. Envelope solutions, such as Huygen wavelets and propagating discontinuities (called signals) are classic examples of solution non uniqueness.

13. Cartan's Magic formula, in terms of the Lie differential acting on exterior differential 1-forms establishes the long sought for combination of dynamics and thermodynamics, enabling non equilibrium systems and many irreversible processes to be computed in terms of continuous topological evolution, without resort to probability theory and statistics.

14. The Lie differential acting on differential forms is not necessarily the same as a linear affine covariant differential acting on differential forms. It is possible to demonstrate that if the process is locally adiabatic (no heat flow in the direction of the evolutionary process), then the Lie differential and the covariant differential can be made to coincide, as they both satisfy the Koszul axioms for an affine connection. This is a surprising result, for, when the argument is reversed, the theorem implies that the ubiquitous affine covariant differential of tensor analysis,

acting on a 1-form of Action, can *always* be cast into a form representing an *adiabatic* process. However, such adiabatic processes need not be reversible.

15. The Lie differential can describe evolutionary processes which are not C2 differentiable, leading to a better understanding of wakes and shocks. On odd dimensional spaces, sequential C1 (translational) processes can be thermodynamically reversible, while intransitive C2 processes (rotation and expansion with a fixed point) can be thermodynamically irreversible.

16. If the evolutionary process described by the Lie differential, affine equivalent or not, leaves the 1-form of Action invariant, then the process is thermodynamically reversible. If the affine covariant differential of tensor analysis induces parallel transport (the covariant differential is zero), then the affine process is adiabatic and reversible.

17. On spaces of Pfaff topological dimension 4, the Cayley-Hamilton theorem produces a characteristic polynomial with similarity invariant coefficients which will generate the format of the Gibbs function for a (universal) van der Waals gas, with a well defined critical point and binodal and spinodal lines. The same technique can be applied to dynamical systems.

The combined thermodynamic-topological perspective presented herein uses the mathematical tools of exterior differential forms to describe the topological features of physical systems, and vector fields to describe the continuous evolutionary processes that may or may not change the topology of the physical system. Examples will demonstrate that topological change is a necessary condition for thermodynamic irreversibility.

## 4.5 Topological properties vs. Geometrical properties

The idea that the presence of a physical system establishes a *topological structure* on a base space of independent variables is different from, but similar to, the geometric perspective of general relativity, whereby the presence of a physical system is presumed to establish a *metric* on a base space of independent variables. The topological features of the physical system are presumed to be encoded in terms of exterior differential forms, which - unlike tensors - are functionally well behaved with respect to differentiable maps that are not invertible. Note that a given base may support many different topological structures; hence a given base may support many different physical systems. In particular, the topology associated with a 1-form of Action need not be the same as the topology associated with the 1-form of Heat, $Q$, or the 1-form of Work, $W$, even though the base variables are the same for each 1-form. The Pfaff topological dimension can be different for each of the 1-forms.

For maps, between base sets, that are C1 differentiable†, but are not invertible, it is *impossible* to predict uniquely the functional forms of covariant or contravariant vector fields, constructed over a final base set, in terms of functional forms given on an initial base set [172]. Point-wise (numeric) values of the tensor fields in certain cases may be predicted, but the functional forms describing neighborhoods are never predictable with respect to such non invertible maps. Hence, classical theories based on tensor fields, which can describe geometrical evolution, will fail to describe topological evolution. It may be surprising to note that (with respect to non invertible, non homeomorphic, maps) it is possible to retrodict the functional forms of covariant vectors and contravariant vector densities on the initial base set in terms of the given functional forms on the final base set. For differentiable evolutionary processes that are diffeomorphisms, topology does not change and both prediction and retrodiction of tensor fields is possible. For differentiable evolutionary processes which are not homeomorphisms, topology changes, and deterministic prediction fails, but deterministic retrodiction remains possible. Hence the feature of topological evolution imposes a sense of asymmetry with respect to an evolutionary parameter - the arrow of time is an artifact of topological change.

Although C1 non invertible maps are not homeomorphisms, and therefore the topology of the initial state and the topology of the final state are not the same, such maps can be continuous. Continuous topological evolution is not an oxymoron, for topological continuity is defined such that the limit points of every subset in the domain (relative to the topology on the initial state) permute into the closure of the subsets in the range (relative to the topology on the final state). The initial and final state topologies need not be the same!

A physical property which is independent of continuous deformation, and is independent from geometric concepts of size and shape, is a primitive example of a topological property. However, not all topological properties fit this useful, but imprecise, description. As examples, note that the number of holes in a rubber sheet is a topological property, and is independent of the continuous deformation of the rubber sheet into different sizes and

†C1 implies connected, where C2 implies smooth.

shapes. The Planck black body radiation distribution of a hot body is a topological property, for the distribution of radiation frequencies (in first approximation) depends only on temperature, but not on the size and shape of the heated sample. Deformation invariants often can be encoded in terms of multi-dimensional integrals. As the elements of the integrand and the integration chain evolve, the value of the integral may be an evolutionary invariant, even though the domain and integration chain are deformed by the evolutionary process. Of special interest are those integral deformation invariants where the integration chain is a closed cycle. Such objects lead to topological "quantum-like" concepts, for the values of the integrals of closed, but not exact, exterior differential forms over different cycles have (by deRham's theorems) rational ratios. (In this volume, quantum effects are discussed only in terms of their topological origin and deRham period integrals.) If the evolutionary process causes the topological quantum number to change, then the process describes a topological quantum transition. Surprisingly, processes of topological evolution can change topology in a continuous manner. A soap film connected to a double loop of wire will form the non orientable surface of a Moebius band. Deformation of the wire into a single loop will cause the soap film to form a disk surface which is orientable. The topological property of orientability has changed continuously in terms of the process described.

More precisely, a topological property is defined as an invariant of a homeomorphism. A homeomorphism is a map from initial to final state, which is continuous and has a continuous inverse. If the homeomorphism is C1 differentiable both ways, then the map is called a diffeomorphism. Diffeomorphisms are the transformations used to define tensors and most geometric properties. Invariance with respect to diffeomorphisms is a constraint employed in many physical theories which are based upon tensor calculus and the calculus of variations. Recall Klein's concept of a (euclidean) geometric property as being defined in terms of the invariants of rotations and translations (which are diffeomorphisms). Yet diffeomorphisms are specialized homeomorphisms which preserve topology. It follows that tensor analysis, so useful in studying geometric concepts, cannot be used effectively to describe topological change, and therefor tensor analysis is inadequate to describe irreversible evolution, where topological change is a necessary condition. However, continuous C1 processes need not be homeomorphisms, and therefor can be used to describe topological change. Exterior differential forms are mathematical objects that are well behaved in a retrodictive sense with respect to functional substitution of C1 continuous, but not invertible, maps; tensor fields are not. It follows that Cartan's exterior differential forms become the mathematical objects of choice for describing continuous topological evolution, and therefor Cartan's mathematics is the mathematics of choice for a theory of Irreversible Thermodynamic processes.

A key topological property is that of dimension. However, the concept of topological dimension is somewhat different from the concept of geometrical dimension. For purposes of the theory developed herein, the topological structure imposed upon a base variety of m independent variables can be used to determine the "Pfaff topological dimension", $n$, which is to be distinguished from the "geometric dimension" of the base variety, $n \leq m$. The primary feature of a topological structure is that it can be used to determine when an evolutionary process involving topological change (such as the change in topological dimension) is continuous. Topological change can occur both continuously and discontinuously. However, in this article, the focus is on *continuous* topological evolution. Herein it will be demonstrated that thermodynamic "relaxation" from some initial configuration to a state of "equilibrium" can be described by a sequence of continuous processes that cause the topological dimension to change from some initial value $n$ to a final value $n \leq 2$.

## 4.6 Pfaff Topological Dimension

Perhaps one of the most important topological tools to be used within the theory of continuous topological evolution is the concept of Pfaff topological dimension. The maximum Pfaff dimension is equal to number of independent variables in the base variety, which in this article has been limited (in most cases) to $n = 4$. For a given 1-form of Action, $A = A_k(x, y, z, t)dx^k$ defined on the base variety of $\{x, y, z, t\}$, it is possible to ask what is the irreducible minimum number of independent functions $\theta(x, y, z, t)$ required to describe the topological features that can be generated by the specified 1-form, $A$. This irreducible number of functions is defined herein as the "Pfaff topological dimension" of the 1-form, $A$. For example, if

$$A = A_k dx^k \Rightarrow d\theta(x, y, z, t)_{irreducible}, \tag{4.6}$$

$$\text{such that } A_k = \partial\theta(x, y, z, t)/\partial x^k, \tag{4.7}$$

then only one function $\theta(x, y, z, t)$ is required to describe the Action, not four. In this example the irreducible Pfaff topological dimension of the 1-form, $A$, is 1, although the geometric dimension of base variety is 4. In a sense, the Pfaff topological dimension defines the existence of a domain of "topological" base variables (topological coordinates) as submersions from the original base variety (geometric coordinates) to the irreducible base variety (topological coordinates). Differential forms constructed on the irreducible base variety of functions, are functionally well defined on the original base variety. (See Chapters 4, 5 and 6)

Relative to the Cartan topology [11], the "Pfaff topological dimension" can be generated by each of the Pfaffian forms associated with each discipline. The irreducible Pfaff topological dimension for any given 1-form $A$ is readily computed by constructing the Pfaff sequence of forms:

$$\textbf{Pfaff sequence} \; :\{A, dA, A\hat{}dA, dA\hat{}dA\}. \tag{4.8}$$

The Pfaff topological dimension is equal to the number of non zero terms in the Pfaff sequence. For example, if the Pfaff sequence for a given 1-form $A$ is $\{A, dA, 0, 0\}$ in a region $U \subset \{x, y, z, t\}$, then the Pfaff topological dimension of $A$ is 2 in the region, $U$. The 1-form $A$, in the region $U$, then admits description in terms of only two, but not less than 2, independent variables, say $\{u^1, u^2\}$. For a differentiable map $\varphi$ from $\{x, y, z, t\} \Rightarrow \{u^1, u^2\}$, the exterior differential 1-form defined on the target variety $U$ of 2 pre-geometry dimensions as

$$A(u^1, u^2) = A_1(u^1, u^2)du^1 + A_2(u^1, u^2)du^2, \tag{4.9}$$

has a functionally well defined pre-image $A(x, y, z, t)$ on the base variety $\{x, y, z, t\}$ of 4 pre-geometric dimensions. This functionally well defined pre-image is obtained by functional substitution of $u^1, u^2, du^1, du^2$ in terms of $\{x, y, z, t\}$ as defined by the mapping $\varphi$. The process of functional substitution is called the pull-back.

$$A(x, y, z, t) = A_k dx^k = \varphi^*(A(u^1, u^2)) = \varphi^*(A_\sigma du^\sigma) \tag{4.10}$$

It may be true that the functional form of $A$ yields a Pfaff topological dimension equal to 2 globally over the domain $\{x, y, z, t\}$, except for sub regions where the Pfaff dimension of $A$ is 3 or 4. These sub regions represent topological defects in the almost global domain of Pfaff dimension 2. Conversely, the Pfaff dimension of $A$ could be 4 globally over the domain, except for sub regions where the Pfaff dimension of $A$ is 3, or less. These sub regions represent topological defects in the almost global domain of Pfaff dimension 4. Applications of both viewpoints will be described below. The important concept of Pfaff topological dimension also can be used to define equivalence classes of physical systems and processes.

The concept defined herein as the "Pfaff topological dimension" was developed more than 110 years ago (see page 290 of Forsyth [68] ), and has been called the "class" of a differential 1-form in the mathematical literature. The term "Pfaff topological dimension" (instead of class) was introduced by the present author in order to emphasize the topological foundations of the concept. More recent mathematical developments can be found in Van der Kulk [219]. The method and its properties have been little utilized in the applied world of physics and engineering. Of key importance is the fact that the non zero existence of the 3-form $A\hat{}dA$, or

$$\textbf{Topological Torsion} = A\hat{}F \tag{4.11}$$

implies that the Pfaff topological dimension of the region is 3 or more, and the non zero existence of the 4-form of *Topological Parity*, $dA\hat{}dA = F\hat{}F$ implies that the Pfaff topological dimension of the region is 4. Either value is an indicator that the physical system (in the sub region) is NOT in thermodynamic equilibrium. It is also important to recall that non zero values of Topological Torsion imply that the Frobenius unique integrability Theorem for the Pfaffian equation, $A = 0$, fails. The concept of *topological parity*, $F\hat{}F$, has its foundations in the theory of Pfaff's problem, with a recognizable 4 dimensional formulation appearing in Forsyth [68] page 100. On a variety of 4 variables, the coefficient of the 4-form $F\hat{}F$ will be defined as the topological parity (or orientation) function, $K$, such that

$$\textbf{Topological Parity} \quad F\hat{}F = K dx\hat{}dy\hat{}dz\hat{}dt = K\Omega_4. \tag{4.12}$$

It is possible to ascribe the idea of entropy production (due to bulk viscosity) to the coefficient $K$ of the Parity 4 form.

The idea of *Topological Torsion*, $A\hat{}F$, has been associated with the idea of magnetic helicity density, a concept that apparently had its electromagnetic genesis with the study of plasmas in WWII. However, the concept of helicity density is but one component of the four dimensional *Topological Torsion 4 vector.*

Recall that a space curve with non zero Frenet - Serret torsion does not reside in a two dimensional plane. non zero Frenet - Serret torsion of a space curve is an indicator that the *geometrical* dimension of the space curve is at least 3. The fact that the Pfaff *topological* dimension of the 1-form, $A$, is at least 3, when $A\hat{}F$ is non zero, is the basis of why the 3-form, $A\hat{}F$, was called "Topological Torsion". The idea of non zero 3-form $A\hat{}F$ also appears in the theory of the Hopf Invariant [26].

The concept of $A\hat{}F$ has also appeared in the differential geometry of connections, where a matrix valued 3-form is known as the Chern-Simons 3-form. However, on varieties without connection or metric, the Chern-Simons concept is not well defined, but the Topological Torsion concept exists and is acceptable, for it does not depend upon the geometric features of metric and/or connection. The concepts can be extended to "pre-geometrical", and therefor topological, domains of dimension greater than 4. Pre-geometry implies that constraints of metric or connection have not been (necessarily) imposed on the base variety.

It is possible to define a "curvature" dimension (at a point) in terms of the number of non null eigenvectors of the Jacobian matrix built from the partial derivatives of the C1 functional components that define the 1-form of Action. The "Curvature" dimension is always less the dimension of the base variety. The implication is that the determinant of the shape matrix is zero. It is possible that the Pfaff topological dimension can exceed the "curvature" dimension.

## 4.7 Evolutionary Invariants

Evolutionary invariants are generally those properties of physical systems that are observables in the sense of physical measurements. Invariants of continuous processes are included in the set of topological properties (invariants of homeomorphisms), and topological properties are included in the set of geometric properties (invariants of diffeomorphisms).

### *4.7.1 Deformation Invariants as Topological Properties*

Topological properties are defined as invariants with respect to homeomorphisms. A more mundane definition is that a topological property is an invariant of a continuous deformation. Certain integral properties of a thermodynamic system are deformation invariants with respect to those continuous evolutionary processes that can be described by a singly parameterized vector field. For the example of an electrodynamic thermodynamic system, the absolute deformation invariants lead to fundamental topological conservation laws, described in the physical literature of electromagnetism as the conservation of charge and the conservation of flux.

Recall the definitions used to describe processes of continuous topological evolution.

**Definition** *A continuous process is defined as a map from an initial state of topology* $T_{initial}$ *into a final state of perhaps dif-*

*ferent topology $T_{final}$ such that the limit points of the initial state are permuted among the limit points of the final state (see p. 97 et.seq. [126]). If the ordering of the limit points is invariant, the process is uniformly continuous. If the ordering (as in a folding of a boundary) or the number of the limit sets is changed, then the process is non uniformly continuous.*

A simple description of a topological property (invariant of a homeomorphism) is an object that is a deformation invariant. Consider a rubber sheet with three holes. Stretch the rubber sheet. The holes may be deformed but the fact that there are 3-holes stays the same under small deformations. The concept of three holes is a topological property. It is remarkable that such topologically coherent objects (the holes) can be determined from those open and closed integrals which are deformation invariants.

A topological deformation invariant is defined as an integral of an exterior differential p-form over a p dimensional manifold, or cycle, *zpd*, such that the Lie differential of the integral of the p-form $\omega$ with respect to a singly parameterized vector field, $\rho V^k$, vanishes, for any choice of deformation parameter, $\rho$.

$$\textbf{Integral Deformation Invariant}: \quad L_{(\rho V^k)} \int_p \omega = 0 \quad any\ \rho \qquad (4.13)$$

The requirements that a given p-form becomes a deformation invariant (and therefor a topological property, invariant with respect to homeomorphisms) is expressed in terms of certain topological constraints. Those objects that remain the same under continuous deformation represent topological, not geometric, properties. However, if the topological constraints required for continuous deformation are not satisfied, then topological change takes place. Topological change would require that the number of holes in the thin rubber sheet example were to change. Topological change can occur continuously or discontinuously. The focus in this article is on continuous topological change, and as will be demonstrated below, topological change is a necessary requirement for thermodynamic irreversibility [172].

### *4.7.2 Absolute Integral Invariants*

There are two types of invariant integrals, Absolute and Relative integral invariants. If the exterior p-form that forms the integrand is exact, the Absolute integral invariant places conditions only on the boundary of the domain of integration. It is these types of objects (Absolute integral invariants) that give a formality to those thermodynamic concepts whereby a physical system reaches equilibrium uniformly within its interior, and yet may couple with its exterior environment via fluxes across its boundary. In such cases, only effects related to the boundary are of consequence. For example, consider physical systems that can be defined by a 1-form of Action, $A$, such that the derived 2-form $F = dA$, is exact. It follows from Stokes theorem that the 2-dimension integral of $F$ is an absolute integral deformation invariant with respect to *all* continuous processes that can be defined by a singly parameterized vector field, subject to a boundary condition that the net flux, $i(\rho V^k)F$, of $F$, across the 1-dimensional boundary of $M$ is zero:

$$L_{(\rho V^k)} \int\int_M F = \int\int_M i(\rho V^k)dF + \int\int_M d(i(\rho V^k)F) \quad (4.14)$$

$$= 0 + \int_{boundary\ of\ M} i(\rho V^k)F \Rightarrow 0. \quad (4.15)$$

This concept is at the basis of the Helmholtz theorems of vorticity conservation (or angular momentum per unit mass) in hydrodynamics, and the conservation of flux in classical electromagnetism. Herein, this concept of deformation invariance of a topologically coherent structure will be written in the form of an exterior differential system [30], $F - dA = 0$. The exterior differential system is to be recognized as topological constraint. From Stokes theorem, the 2 dimensional domain of finite support for $F$ can not, in general, be compact without boundary, unless the Euler characteristic vanishes. There are two exceptional cases for absolute invariance of the integral, and they occur when the integration domain is compact without boundary. Such two dimensional domains which have a zero Euler characteristic are the torus and the Klein-Bottle, but these situations require the additional topological constraint that $F\hat{}F \Rightarrow 0$. The fields in these exceptional cases must reside on these exceptional compact surfaces without boundary, which form topological coherent structures. Note that an evolutionary process could start with $F\hat{}F \neq 0$, and possibly evolve to a state with $F\hat{}F = 0$. If such residue states are compact without boundary, then they must be either tori or Klein bottles.

The same integration technique can be applied to non exact but closed p-forms.

### *4.7.3 Relative Integral Invariants*

If the integration of the exact 2-form, $F$, is over a *closed* two dimensional integration chain, designated as a 2 dimensional cycle, z2d (which may or may not be a 2 dimensional boundary), then the Integral is invariant for any deformation factor, $\rho$ :

$$L_{(\rho V^k)} \int\int_{z2d} F = \int\int_{z2d} i(\rho V^k)dF + \int\int_{z2d} d(i(\rho V^k)F) = 0 + 0. \quad (4.16)$$

The two integrals on the right vanish, the first due to the fact that $dF = 0$, and the second due the fact that the closed integral over an exact form vanishes. Close integrals of exact p-forms are always relative deformation integral invariants. However, the same technique can be applied to non exact but closed p-forms. For electromagnetism, there are several exact p-forms, each producing a relative deformation integral invariant. For example, the 3-form of charge-current density is exact, $J = dG$. The 4-forms that define the Poincare Invariants are exact: $F\hat{}F = d(A\hat{}G)$ and $F\hat{}G - A\hat{}J = d(A\hat{}G)$. See Section 4.5.

If the conditions of relative integral invariance are applied to an arbitrary 1-form of Action, then the relative integral invariance condition becomes

$$L_{(\rho V^k)} \int_{z1d} A = \int_{z1d} i(\rho V^k)dA + \int_{z1d} d(i(\rho V^k)A) \quad (4.17)$$

$$= \int_{z1d} i(\rho V^k)F + 0 \Rightarrow 0. \quad (4.18)$$

It follows the $i(\rho V^k)dA$ must be zero on the cycle $z1d$ for any deformation parameter $\rho$. Cartan has shown that this is the condition that implies the process $\rho V^k$ has a "Hamiltonian" representation [38]. See Section 3.1.1

### *4.7.4 Holder Norms, Period Integrals and Topological Quantization*

Besides the invariant structures considered above, the Cartan methods may be used to generate other sets of topological invariants. Realize that over a domain of Pfaff dimension $n$ less than or equal to $N$, the Cartan criteria admits a submersive map to be made from N to a space of minimal dimension n. Assume the submersive map produces functions

$$[V^1(x,y,z..), V^2(x,y,z..), ...V^n(x,y,z..)] \quad (4.19)$$

with a differential volume element, $\Omega_n = dV^1\hat{}dV^2\hat{}...\hat{}V^n$. Then construct the n-1 form,

$$C = i(V^1, V^2, ..., V^n)\Omega_n \quad (4.20)$$

Define an integrating factor $\rho$ in terms of the Holder norm,

$$\rho = 1/\lambda = 1/\{a(V^1)^p + b(V^2)^p + c(V^3)^p + .....\}^{m/p}. \tag{4.21}$$

Then multiply the $C$ form by $\rho$ to produce n-1 form density (current) $J$ as:

$$J = i(V^1, V^2, V^3, ...)Vol = \\ \rho\{V^1\ dV^2\hat{}dV^3... \ - V^2\ dV^1\hat{}dV^3... + V^3\ dV^1\hat{}dV^2... - ...\}. \tag{4.22}$$

**Theorem 1.** *The n-1 form J is closed, for any choice of constants a,b,c.. and for any p, if the holder homogeneity index* $m = n: \ dJ = 0$.

It is remarkable that the "current" $J$ so defined has a vanishing exterior differential, independent of the value of p for a given m (equal to the dimension of the volume element), and for all values of the constants, plus or minus a,b,c...). All such "currents" thereby define a "conservation law". As the map defining the components of the vector field in terms of the base {x,y,z..} is presumed to be differentiable, then the n-1 form, $J$, has a well defined pull back on the base space (almost everywhere), and its exterior differential on the base space also vanishes everywhere mod the defects. That is, the form $J$ is locally exact. The number of negative coefficients in set $\{a, b, c, d...\}$ determines the signature index of the Holder norm. The number $m$ determines the homogeneity index. The Holder integrating factors are more familiar when $m = 1,\ p = 1$ which generates the barycentric coordinates $\{a, b, c, d...\}$ of Moebius, [27], and for $m = 1,\ p = 2,\ a = b = c...$, which is known as the Gauss map. Use for both of these special Holder constructions will be developed in that which follows.

The integrals of these closed currents, when integrated over closed N-1 dimensional chains, form deformation invariants, with respect to any evolutionary process that can be described by a vector field, for

$$L_{(\rho\mathbf{V})} \int_{z(n-1)d} J = \int_{z(n-1)d} i(\rho\mathbf{V})dJ + \int_{z(n-1)d} d(i(\rho\mathbf{V}))J) = 0 + 0 = 0. \tag{4.23}$$

These integral objects appear as "topological coherent" structures (which may have defects or anomalous sources, when the integrating factor $1/\lambda$ is not defined). The integration chain is a $(n-1)$ dimensional $(d)$ cycle $z$.

The compliment to the zero sets of the function $\lambda$ determine the domain of support associated with the specified vector field. The closed n-1 form, $J$, that satisfies the conservation law, $dJ = 0$, has integrals over closed domains that have rational fraction ratios. As this n-1 current is closed globally, it may be deduced on a connected local domain from a n-2 form, $G$. In every case $J$ has a well defined pull-back to the base variety, x,y,z,t. Note that the n functions $[V^x(x, y, z..), V^y(x, y, z..), V^z(x, y, z..), ...]$ represent the minimum number of Clebsch variables that are equivalent to the original action, $A$, over the domain of support. As each of these integrals is intrinsically closed, the Lie differential with respect to any C1 vector field, $\rho\mathbf{V}$, is a perfect differential, such that (when integrated over closed domains that are p-1 boundaries) the evolutionary variation of these closed integrals vanishes. These n-1 integrals are relative integral invariants for any C1 evolutionary processes, or flows. The values of the integrals are zero if the closed integration domains are boundaries, or completely enclose a simply connected region. If the closed integration domains encircle the zeros of the function $\lambda$, then the values of the integrals are proportional to the integers; i.e., their ratios are rational.

In general, by deRham's theorems, these values of these period integrals, for different closed integration chains in domains where $dJ = 0$, have rational ratios [173]. When the evolution of a period integral is such that the integer changes, the process can describe the decay from a quantized stationary state of topological quantum number m to a state of topological quantum number n:

$$\textbf{Topological Quantization}: \; L_{(\rho\mathbf{V})} \int_{z(n-1)d} J = n \text{ constant.} \tag{4.24}$$

Note that each signature of $\lambda$ must be investigated. For the elliptic (positive definite) signature, the singular points are the stagnation points, and the domain of support excludes those singularities. For the hyperbolic signatures, the domain of support excludes the hyperbolic singularities of lower dimension, such as the light cone. Further note that a given vector field may not generate real domains of support for all possible signatures of the quadratic form, $\lambda$.

Details and applications of homogeneous constructions that give rise to period integrals are presented in Chapter 6.

## 4.8 Unique Continuous Evolutionary Processes

Evolutionary processes can preserve the topological properties of a physical system, or they can change them. In this monograph, those processes which can change continuously the topology of a physical system are of major interest, for topological change is a necessary requirement of thermodynamic irreversibility. Intuitively, a process applied to a physical system can be arbitrary, which implies that the process $V$ is not necessarily dictated by the topological structure of the physical system, $A$. This intuitive idea is not precise. In contrast, for a given physical system, $A$, there are two vector direction fields that are determined *uniquely* from the functions that defined the topological structure of the physical system. One of these vector fields, $\mathbf{V}_E$, is uniquely defined on contact manifolds of odd topological dimension, $2n+1$. $\mathbf{V}_E$ is called an "extremal" field. The second vector direction field, $\mathbf{V}_T$, is defined uniquely on symplectic manifolds of even topological dimension, $2n+2$. $\mathbf{V}_T$ is called a "torsion" vector. Recall that the topological dimension can be smaller than or equal to the geometric dimension. The concept that these two vectors are uniquely determined implies that the representation is in terms of topological, not geometrical, coordinates.

The extremal vector $\mathbf{V}_E$ on a contact manifold is proportional to the unique null eigen value of the $2n+1 \times 2n+1$ anti-symmetric matrix of functions that forms the components of the 2-form, $dA$. Such null eigen vectors exist (uniquely) only on *topological* domains of odd maximal rank, typically in this monograph, equal to 3. The extremal vector,$\mathbf{V}_E$, satisfies the equations:

$$\textbf{Extremal Vectors} \quad : \quad i(\mathbf{V}_E)dA = 0, \tag{4.25}$$

$$\Omega = dx^1\hat{}dx^2\hat{}...dx^{2n+1} \tag{4.26}$$

$$L_{(\mathbf{V}_E)}A = d(i(\mathbf{V}_E)A). \tag{4.27}$$

It will be demonstrated in the subsequent chapters that extremal vector fields admit a Hamiltonian representation, or generator, that describes the evolutionary direction field. Extremal Hamiltonian vector fields (without fixed points) do not alter the topological properties of the physical system represented by the 1-form, $A$. On geometric domains of dimension greater than 2n+1, there can exist are other vector direction fields that have a Hamiltonian (or better said a Bernoulli-Casimir) generator. However, these "Hamiltonian" direction fields are not uniquely defined in terms of the functions that define the topology of the physical system. For example, on a geometric domain of 2n+2 dimensions, if there exists one null eigenvector of the 2-form, $dA$, then there must exist at least two null eigenvectors. The differences between extremal Hamiltonian processes and the Bernoulli processes will be discussed in the subsequent chapters.

On the other hand, on *topological* domains where $dA$ is of even maximal rank, extremal vectors (null eigenvectors relative to $dA$) do not exist. Yet there is a unique vector, the vector, $V_T$, that is determined by the topological structure of the physical system, $A$. Such unique vectors exist only on topological domains of even dimensional maximal rank, typically in this monograph, equal to 4. The Topological Torsion vector satisfies the equations:

$$\textbf{Topological Torsion Vector} \quad : \quad i(\mathbf{V}_T)\Omega = A\hat{}dA...dA, \quad (4.28)$$

$$\Omega = dx^1\hat{}dx^2\hat{}...dx^{2n+2} \quad (4.29)$$

$$L_{(\mathbf{V}_T)}A = i(\mathbf{V}_T)dA = \sigma A \quad (4.30)$$

$$i(\mathbf{V}_T)A = 0 \quad (4.31)$$

These equations will be developed in detail in the next section. It will also be shown that an evolutionary process which has a component in the direction of the "Topological Torsion" vector will produce a thermodynamically irreversible process on the physical system defined by the 1-form of Action, $A$. Evolution in the direction of the Topological Torsion vector causes topological change.

The previous paragraph contains the first introduction to the concept of topological torsion to appear in this monograph. The properties of the topological torsion vector will be extolled and examined again and again in the chapters that follow. The existence of a Topological Torsion direction field is a signal that the physical system is not in equilibrium. Evolution in the direction of the Topological Torsion vector is thermodynamically irreversible if the divergence of $\mathbf{V}_T$ is not zero. The dogmatic insistence on topological invariance in many classical physical theories in effect excludes the concept of the "topological torsion". When $\sigma = 1$, the Topological Torsion vector has been called the "Liouville vector field" (see page 65 [123] [124]). If the topological structure of the physical system, $A$, evolves (or decays) from a domain where the rank of $dA$ is even to a domain where the maximal rank of $dA$ is odd, then the components of $\mathbf{V}_T$ become proportional to a characteristic direction field, which has both extremal properties and associated properties.

The extremal processes, $\mathbf{V}_E$ , can always be put into correspondence with a Hamiltonian process, but those processes represented by a direction field component proportional to $V_T$ do not have a Hamiltonian representation, unless the divergence of $V_T$ is zero. These features apply not only to topological dimension 3 and 4, but are also valid for topological dimensions which are odd ($V_E$ for $n = 2k+1$) or even ($\mathbf{V}_T$ for $n = 2k+2$). The topological refinement induced by the process forms two categories related to a Contact structure ($n = 2k+1$) or to a Symplectic structure ($n = 2k+2$). Note that the concept of "uniqueness" relates to the direction field $V$ that represents a process, but such direction fields as a vector field are unique only to within an arbitrary (deformation) factor.

Continuous evolution includes two equivalence classes of processes: those processes that preserve topological features (homeomorphisms) and those processes that do not ( non homeomorphisms). The latter class is the class of processes that describe continuous topological evolution, and it is this class which is studied extensively in this article. As will be demonstrated below, the topological structure of a physical system leads to the consideration of odd dimensional integrals of the type $\int_{2k+1} A\hat{}dA...$ and even dimensional integrals of the type $\int_{2k+2} dA\hat{}dA....$ If these integrals are deformation invariants they represent a topological property that is an evolutionary invariant. Of particular interest is the set of even and odd dimensional integrals where the integration chain is a closed cycle.

The class of continuous processes that describe topological change can be divided into two distinct classes, A and B.

- Class A. This equivalence class of non homeomorphic continuous processes preserves the even dimensional integrals as deformation invariants, but causes the values of the odd dimensional integrals to change. The Helmholtz conservation of vorticity concept is a classic example of when an even dimensional topological property is preserved. Such processes will be called Helmholtz (or Symplectic) processes, in general, when the 2-form of Action, $dA$, is an evolutionary invariant. The Poincare integral invariants of classical mechanics are further examples of even dimensional integral invariants. Extremal and Hamiltonian processes are special cases of Helmholtz processes. However, it will be demonstrated that all such Helmholtz processes, which can produce topological change of the odd dimensional integrals, are thermodynamically reversible. Topological change is a necessary, but not sufficient, condition for continuous thermodynamic irreversibility.

- Class B. This equivalence class of non homeomorphic continuous processes causes the values of both the even and the odd dimensional integrals to change. Both the odd and the even topological features of the physical system are modified. It is this equivalence class that contains those processes which are thermodynamically irreversible. Without being too precise, both energy and angular momentum must change if a process is to be thermodynamically irreversible. Pasting together is a continuous process for which the topology of the final system state is not necessarily the same as the topology of the initial system state. Separation or cutting into parts is a discontinuous process for which the system topology of the final state is not the same as the system topology of the initial state. The obvious topological property that changes is the number of parts. Projections from higher dimensions to lower dimensions are classic examples of many to one differentiable maps that are not invertible. The obvious topological property that changes is the property of dimension. Consider a flat putty disc in the shape of an annulus. Deform the putty continuously such that the points that make up the central hole are pasted together. On the other hand make an interior cut in a disk of putty and discontinuously separate the points to make a hole. The obvious topological property that changes is the number of holes. (Discontinuous processes are more or less ignored in this presentation.)

In this article, attention will be focused on dissipative turbulent systems with thermodynamic irreversible processes such that the Pfaff topological dimensions of $A$, $W$, and $Q$ will be maximal and equal to 4. (The techniques can be extended to higher dimensional geometric spaces.) These Turbulent systems of Pfaff dimension 4 are not topologically equivalent to Equilibrium systems (for which the topological dimension is 2, at most). Topological defects in the Turbulent state will be associated with sets of space time where the Pfaff topological dimensions of $A$, $W$, and $Q$ are not maximal. It is remarkable that such topological defect sets can form attractors causing self organization and long lived states of Pfaff dimension 3, which are far from equilibrium. Examples will be presented below.

### *4.8.1 Physical Systems: Equilibrium, Isolated, Closed and Open*

Physical systems and processes are elements of topological categories determined by the Pfaff topological dimension (or class) of the 1-forms of Action, $A$, Work, $W$, and Heat, $Q$. For example, the Pfaff topological dimension of the exterior differential 1-form of Action, $A$, determines the various species of thermodynamic systems in terms of distinct topological categories. The are two topological thermodynamic categories that are determined by the closure (or differential ideal) of the 1-form of Action, $A \cup dA$, and the closure of the 3-form of topological torsion, $A\hat{}dA \cup dA\hat{}dA$. The first category is represented by a connected Cartan topology, while the second category is represented by a disconnected Cartan topology. The Cartan topology is discussed in detail in Chapter 4.

**Connected Topology** $A\hat{}F = 0$

1. Equilibrium physical systems are elements such that the Pfaff topological dimension of the 1-form of Action, $A$, is 1.

2. Isolated physical systems are elements such that the Pfaff topological dimension of the 1-form of Action, $A$, is 2, or less. Isolated systems of Pfaff dimension 2 need not be in equilibrium, but (in historic language) do not exchange radiation or mass with the environment.

**Disconnected Topology** $A\hat{}F \neq 0$

1. Closed physical systems are elements such that the Pfaff topological dimension of the 1-form of Action, $A$, is 3. Closed systems can exchange radiation, but not mass, with the environment.

2. Open physical systems are such that the Pfaff topological dimension of the 1-form of Action, $A$, is 4. Open physical systems can exchange both radiation and mass[‡] with the environment.

$$\textbf{Systems} \; : \; \text{defined by the Pfaff dimension of } A \tag{4.32}$$
$$dA = 0 \quad \textbf{Equilibrium} \text{ - Pfaff dimension 1} \tag{4.33}$$
$$A\hat{}dA = 0 \quad \textbf{Isolated} \text{ - Pfaff dimension 2} \tag{4.34}$$
$$d(A\hat{}dA) = 0 \quad \textbf{Closed} \text{ - Pfaff dimension 3} \tag{4.35}$$
$$dA\hat{}dA \neq 0. \quad \textbf{Open} \text{ - Pfaff dimension 4.} \tag{4.36}$$

Note that these topological specifications as given above are determined entirely from the functional properties of the physical system encoded as a 1-form of Action, $A$. The system topological categories do not involve a process, which is encoded (to within a factor) by some vector direction field, $\mathbf{V}_4$. However, the process $\mathbf{V}_4$ does influence the topological properties of the work 1-form $W$ and the Heat 1-form $Q$. Compare these topological definitions, whereby Equilibrium or Isolated systems are determined in terms of two independent variables at most, and Duhem's theorem

[‡]The use of the word mass to distinguish between closed and open systems is a legacy that ought to be changed to "mole or particle" number, as it is now known that mass energy can be converted to radiation, and radiation can produce massive pairs.

> "Whatever the number of phases, components and chemical reactions, if the initial mole numbers $N_k$ of all components are specified, the equilibrium state of a closed system is completely specified by two independent variables. (p.182 [160])"

*4.8.2 Equilibrium versus non Equilibrium Systems*

The intuitive idea for an equilibrium system comes from the experimental recognition that the intensive variables of Pressure and Temperature (conjugate to volume and entropy) become domain constants in an equilibrium state: $dP \Rightarrow 0,\ dT \Rightarrow 0.$ A definition made herein is that the Pfaff topological dimension in the interior of a physical system which is in the equilibrium state is at most 1 [12]. Formally, the idea is restated such that the equilibrium state is a Lagrangian submanifold of a 4 dimensional symplectic manifold, and upon this Lagrangian submanifold, the 2-form $dA$, that generates the symplectic structure, vanishes. Hence the equilibrium state is of Pfaff topological dimension 1:

$$\textbf{Equilibrium}\ \ \{A \neq 0,\ dA = 0,\ A\hat{}dA = 0,\ dA\hat{}dA = 0\}.$$

The isolated physical system is of Pfaff dimension 2,

$$\textbf{Isolated}\ \ \{A \neq 0,\ dA \neq 0,\ A\hat{}dA = 0,\ dA\hat{}dA = 0\}.$$

For both the isolated or equilibrium system, the Cartan topology generated by the elements of the Pfaff sequence for $A$ is then a connected topology of one component, as $A\hat{}dA = 0$, (see chapter 4).

Although the Pfaff topological dimension of $A$ is at most 2 in the isolated state, processes in the isolated state are such that the Work 1-form and the Heat 1-form must be of Pfaff dimension 1. For suppose $W = PdV$, then $dW = dP\hat{}dV \Rightarrow 0$ if the pressure is a domain constant. Similarly, suppose $Q = TdS$, then $dQ = dT\hat{}dS \Rightarrow 0$ if the temperature is a domain constant. Hence both $W$ and $Q$ are of Pfaff dimension 1 for this isolated example. If the Pfaff dimension of the 1-form of Action is 1, then $dA \Rightarrow 0$. It follows in this more stringent case that $W \Rightarrow 0$. Hence for elementary systems the Pressure must vanish or the Volume is constant, and the Heat 1-form is a perfect differential, $Q = d(U)$.

Of particular interest herein are those regions of base variables for open, non equilibrium, Turbulent physical systems, formed by the closure[§] of

[§]The closure of the p-form $\Sigma$ is the union of $\Sigma$ and $d\Sigma$, which Cartan has called a differential ideal.

the 3-forms $A\hat{}dA$, $W\hat{}dW$, and $Q\hat{}dQ$. For such regions, the Pfaff topological dimension of the 1-forms, $A$, $W$, and $Q$, are all initially of Pfaff topological dimension 4,

$$dA\hat{}dA \neq 0, \quad dW\hat{}dW \neq 0, \quad dQ\hat{}dQ \neq 0, \tag{4.37}$$

save for defect regions that are of Pfaff dimension 3 (or less). It is remarkable that evolutionary dissipative irreversible processes in such open systems can describe evolution to regions of base variables where the Pfaff topological dimension of the 1-form of Action, $A$, changes from 4 to 3. Such processes describe topological change in the physical system. For a given 1-form of Action, $A$, those regions of Pfaff topological dimension 3, once created, form topological "defect structures" in the closure of the 3-form, $A\hat{}F$. The defect structures of the 1-form of Action, $A$, (of Pfaff dimension 3) can behave as long lived (excited) states of the initial physical system, but they are far from equilibrium and are not isolated, for they are not of Pfaff topological dimension equal to 2 or less. Such excited states (of odd topological dimension) can admit extremal processes of kinematic perfection, and can have a Hamiltonian generator for the kinematics represented as a system of first order ordinary differential equations. The Hamiltonian evolution remains contained in the defect structure, unless topological fluctuations destroy the kinematic perfection.

Such concepts can be applied to a model of cosmology (where the stars are the defect structures), to turbulent plasmas and fluids (where wakes are the defect structures), and to a better understanding of the arrow of time. Although the defects in the Turbulent non equilibrium regime are not necessarily equilibrium structures, once formed and self organized as coherent topological structures of Pfaff dimension 3, they can evolve along extremal trajectories that are not dissipative, Indeed such extremal processes have a Hamiltonian representation. These "stationary", or long lived (excited), states of Pfaff dimension 3, indeed are states "far" from the equilibrium state, which requires a Pfaff dimension of 1. Note that the word "far" does not imply a "distance". The Pfaff dimension 3 and 4 sets are not even "connected" to the equilibrium states in a topological sense. The non equilibrium but isolated states of a physical system that are "near-by" to the equilibrium state, are "connected" to the equilibrium state, and are of Pfaff dimension 2.

The descriptive words of self-organized states far from equilibrium have been abstracted from the intuition and conjectures of I. Prigogine [160]. However the topological theory presented herein presents for the first time a solid, formal, mathematical justification (with examples) for the Prigogine conjectures. Precise definitions of equilibrium and non equilibrium systems, as well as reversible and irreversible processes can be made in terms of the topological features of Cartan's exterior calculus. Thermodynamic irreversibility and the arrow of time are well defined in a topological sense [206], a technique that goes beyond (and without) statistical analysis. Thermodynamic irreversibility and the arrow of time requires that the evolutionary process produce topological change.

### *4.8.3 Change of Pfaff Topological Dimension*

It should be noted that the closed components of the 1-form of Action do not effect the components of the 2-form of intensities, $F = dA = d(A_c + A_0) = 0 + d(A_0) = F_0$. However, these "gauge" additions of closed forms, $A_c$, do influence the topological dimension of the 1-form of Action. For example, let $A_0$ be of Pfaff Topological dimension 2, representing an isolated system where $A_0\hat{}dA_0 = 0$. Then by addition of a closed component to the original action, the new 1-form of Action, $A = A_c + A_0$ could have a topological dimension of 3:

$$A\hat{}dA = (A_c + A_0)\hat{}dA_0 = A_c\hat{}dA_0 \neq 0, \tag{4.38}$$

So the addition of a closed component to the 1-form of Action could change the system from an isolated system of Pfaff dimension 2 to a closed system of Pfaff dimension 3. The 4-form $dA\hat{}dA$ is not influenced by the (gauge) addition to the original 1-form of Action.

$$dA\hat{}dA = dA_0\hat{}dA_0. \tag{4.39}$$

In higher dimension, such gauge additions imply that the Pfaff dimension can change according to the rule, $2n \Rightarrow 2n + 1$.

It is also possible to change the Pfaff dimension of a 1-form by "renormalization", or better said, by "rescaling" with a multiplying function, often in the form of an integrating factor. For example, consider the 1-form $A_0$ of Pfaff dimension 4, such that $d(A_0\hat{}dA_0) \neq 0$. Next rescale the 1-form such that $A = \beta A_0$. Then

$$d(A\hat{}dA) = d(\beta^2 A_0\hat{}dA_0) \Rightarrow 0, \tag{4.40}$$

if $\beta^2$ is an integrating factor for the 3-form $A_0\hat{}dA_0$. In 4 dimensions there exists an infinite number of such functions that serve as integrating factors for the 3-form of Topological Torsion, $A_0\hat{}dA_0$. The integrating factors (which can be formulated from Holder norms) can be interpreted as distributions of "density" which change the Pfaff topological dimension from 4 to 3, or, in general, from $2n + 2 \Rightarrow 2n + 1$. Such distributions can be put into correspondence with "stationary" states far from equilibrium.

As an example of how the Pfaff dimension of a 1-form can be modified by a gauge addition, see section (3.3.3) where a 1-form representing a Bohm-Aharanov-Abrikosov singular "vortex" string,

$$\gamma = b(ydx - xdy)/(x^2 + y^2), \tag{4.41}$$

is added to a 1/r potential for a point source. The bare m/r "Coulomb" potential, $A_0 = m/\sqrt{(x^2 + y^2 + z^2)}dt$ exhibits no Topological Torsion but does exhibit Topological Spin. The 1/r potential term implies that $A_0 \neq 0$. Hence the 1-form of Action representing a bare "coulomb" potential, is not in equilibrium, but does represent a connected "isolated" topology of Pfaff dimension 2. The combined 1-form of Action,

$$A = b(ydx - xdy)/(x^2 + y^2) + m/\sqrt{(x^2 + y^2 + z^2)}dt, \tag{4.42}$$

even though $d\gamma = 0$, is of Pfaff dimension 3, not 2. The Topological Torsion 3-form $A\hat{}F$ depends on both b and m, and is zero if $b = 0$, or if $m = 0$, reducing the Pfaff dimension of the modified 1-form back to 2.

### *4.8.4 Systems with Multiple Components*

One of the most remarkable properties of the Cartan topology generated by a Pfaff sequence is due to the fact that when $A\hat{}dA = 0$, (Pfaff dimension 2 or less) the physical system is reducible to a single connected topological component. This single connected topological component need not be simply connected. The Topological Torsion field vanishes on equilibrium domains.

On the other hand when $A\hat{}dA \neq 0$, (the Pfaff topological dimension of the 1-form, $A$, is 3 or more) the physical system admits more than one topological component (and the topology is a disconnected topology see Chapter 4). The bottom line is that when the Pfaff dimension is 3 or greater (such that conditions of the Frobenius unique integrability theorem are not satisfied), solution uniqueness to the Pfaffian differential equation, $A = 0$, is lost. If solutions exists, there is more than one solution. Such concepts lead to propagating discontinuities (signals), envelope solutions ¶ (Huygen wavelets), an edge of regression (the Spinodal line of phase transitions) a lack of time reversal invariance, and the existence of irreducible affine torsion in

¶See section 6.6

the theory of connections. It is the opinion of this author that a dogmatic insistence that a viable physical theory must give a unique prediction from a set of given initial conditions historically has hindered the understanding of irreversibility and non equilibrium systems. Irreversibility and non equilibrium are concepts that require non uniqueness, and demand that the dogma mentioned above has to be rejected.

## 4.9 Thermodynamic Processes

### *4.9.1 Continuous Processes*

All continuous processes (see Chapter 5) may be put into equivalence classes as determined by the vector direction fields, $V$, that locally generate a flow. For example on a domain of geometric dimension, n, and for the 1-form, $A$, those n-1 vector fields, $\mathbf{V}_{associated}$, that satisfy the transversal equation,

$$\textbf{Associated Class:} \quad i(\rho\mathbf{V}_{associated})A = 0, \tag{4.43}$$

are said to be elements of the associated class of vector fields relative to the form $A$. If the direction field of the 1-form of Action is considered to be a fiber, then the associated vectors are also said to be "horizontal". The associated vectors will form a distribution orthogonal to the 1-form, $A$, but the distribution need not be a smooth foliation. That is, the fiber direction field is not necessarily the normal field to an implicit hypersurface. The requirement for a smooth foliation is that the associated 1-form be of Pfaff topological dimension 2 or less. For such associated processes acting on a 1-form of Action, $A$, the "internal interaction energy" vanishes. As shown below, processes generated by associated vectors relative to the 1-form of Action, $A$, are also included in the set of thermodynamic locally adiabatic processes. Other locally adiabatic processes are generated by those processes which are associated vectors of the exterior derivative of the internal energy, U. In both cases, the adiabatic processes are null vectors of the Heat 1-form, in the sense that $i(\rho\mathbf{V}_{adiabatic})Q = 0$.

Those vector fields, $\mathbf{V}_{extremal}$, that satisfy the equations,

$$\textbf{Extremal Class:} \quad i(\rho\mathbf{V}_{extremal})dA = 0, \tag{4.44}$$

are said to be elements of the extremal class of vector fields. As the matrix of functions that define the 2-form $dA$ is antisymmetric, the extremal vector is proportional to that eigen vector of the antisymmetric matrix that has a zero eigen value. If the matrix $dA$ is of maximum rank, then there is only one (unique) eigen vector with zero eigen value, and that null eigen vector exists, only if the Pfaff topological dimension of the 1-form $A$ is odd $(2n+1)$. In other words, the 2-form $dA$ defines a Contact manifold. The extremal direction field is completely determined (to within a factor) by the component functions of the 1-form $A$ utilized in its definition. Note that the work 1-form $W = i(\rho\mathbf{V}_{extremal})dA \Rightarrow 0$ vanishes for extremal evolutionary processes.

If the Pfaff topological dimension of the 1-form $A$ is even, then a unique extremal vector does not exist. The reduced topological domain (not necessarily the entire geometric domain) is a symplectic manifold of even dimensions, $(2n+2)$. However, on the symplectic manifold of 4 geometric dimensions and 4 topological dimensions, it follows that there does exist a unique vector direction field, the Topological Torsion vector, $\mathbf{V}_{Torsion}$, completely determined (to within a factor) in terms of the functions which define the physical system.

$$\textbf{Topological Torsion Class} \quad : \quad i(\rho\mathbf{V}_{Torsion})dA = \sigma A, \qquad (4.45)$$

$$i(\rho\mathbf{V}_{Torsion})A = 0, \qquad (4.46)$$

In the next section it will be shown that evolution with a component in the direction of the "Topological Torsion" vector will produce an irreversible process on the physical system (as encoded by the Action 1-form), if the divergence of the "Topological Torsion" vector is not zero. This "Topological Torsion" vector equivalent to the 3-form $A\hat{}dA = A\hat{}F$ is always an associated vector, but it is not necessarily an extremal vector, relative to the Action 1-form, $A$. The Torsion vector is identically zero on domains of Pfaff topological dimension 2. Hence non zero values of the Torsion vector are an indication that the physical system, $A$, is not an equilibrium system. The Topological Torsion vector exists only on domains of Pfaff topological dimension 3 or greater, in the same sense that Frenet-Serret torsion exists only on domains of geometric dimension 3 or greater. With respect to evolution in the direction of the Torsion Current, the symplectic 4D volume is contracting or expanding exponentially unless $\sigma = 0$. If the divergence of $\mathbf{V}_{Torsion}$ vanishes, $\sigma \Rightarrow 0$, and therefore such vector fields cannot represent a symplectic process (which preserves the volume element, $dA\hat{}dA$). The factor, $\sigma$, is a Liapunov function and defines the stability of the process (depending on the sign of $\sigma$). When $\sigma = 1$, the Torsion vector has been called the "Liouville vector" [124].

Vector fields which are both extremal and associated are said to be elements of the characteristic class, $\mathbf{V}_{characteristic}$, of vector fields [98].

$$\textbf{Characteristic Class} \quad : \quad i(\rho\mathbf{V}_{characteristic})A = 0, \tag{4.47}$$
$$\textbf{and} \quad : \quad (\rho\mathbf{V}_{characteristic})dA = 0. \tag{4.48}$$

Note that characteristic flow lines generated by $\mathbf{V}_{characteristic}$ of the Characteristic class preserve the Cartan topology, for each form of the Cartan topological base is invariant with respect to the action of the Lie differential to characteristic flows (See Chapter 4). Characteristics are often associated with wave phenomena, and propagating discontinuities. They are locally adiabatic. The Topological Torsion vector mentioned above may have zero divergence on certain geometric subsets of space-time, but these domains are of Pfaff topological dimension 3 (although of geometric dimension 4). In such cases, the Topological Torsion vector will be a characteristic vector for the 1-form of Action, $A$. These and other properties of the "Topological Torsion" vector will be described in detail by examples presented below.

### *4.9.2 Reversible and Irreversible Processes*

The Pfaff topological dimension of the exterior differential 1-form of Heat, $Q$, determines important topological categories of processes. From classical thermodynamics "The quantity of heat in a reversible process always has an integrating factor" [76] [142] . Hence, from the Frobenius unique integrability theorem, which requires $Q\hat{}dQ = 0$, all reversible processes are such that the Pfaff dimension of $Q$ is less than or equal to 2. Irreversible processes are such that the Pfaff dimension of $Q$ is greater than 2, and an integrating factor does not exist. A dissipative irreversible topologically *turbulent* process is defined when the Pfaff dimension of $Q$ is 4.

$$\textbf{Processes} \quad : \quad \textbf{as defined by the Pfaff dimension of } Q$$
$$Q\hat{}dQ = 0 \qquad \textbf{Reversible} \text{ - Pfaff dimension 2} \tag{4.49}$$
$$d(Q\hat{}dQ) \neq 0. \qquad \textbf{Turbulent} \text{ - Pfaff dimension 4.} \tag{4.50}$$

Note that the Pfaff dimension of $Q$ depends on both the choice of a process, $\mathbf{V}_4$, and the system, $A$, upon which it acts. As reversible thermodynamic processes are such that $Q\hat{}dQ = 0$, and irreversible thermodynamic processes are such that $Q\hat{}dQ \neq 0$, Cartan's formula of continuous topological evolution can be used to determine if a given process, $\mathbf{V}_4$, acting on a physical system, $A$, is thermodynamically reversible or not:

$$\begin{array}{rcl} \textbf{Processes defined by} & : & \textbf{the Lie differential of } \mathrm{A} \\ L_{(\rho\mathbf{V}_4)}A & = & Q \qquad (4.51) \\ \textbf{Reversible Processes } \rho\mathbf{V}_4 & : & Q\hat{}dQ = 0, \qquad (4.52) \\ L_{(\rho\mathbf{V}_4)}A\hat{}L_{(\rho\mathbf{V}_4)}dA & = & 0, \qquad (4.53) \\ \textbf{Irreversible Processes } \rho\mathbf{V}_4 & : & Q\hat{}dQ \neq 0, \qquad (4.54) \\ L_{(\rho\mathbf{V}_4)}A\hat{}L_{(\rho\mathbf{V}_4)}dA & \neq & 0. \qquad (4.55) \end{array}$$

Remarkably, Cartan's magic formula can be used to describe the continuous dynamic possibilities of both reversible and irreversible processes, acting on equilibrium or non equilibrium systems, even when the evolution induces topological change, transitions between excited states, or changes of phase, such as condensations.

It is important to note that the direction field, $\mathbf{V}_4$, need not be topologically constrained such that it is singularly parameterized. That is, the evolutionary processes described by Cartan's magic formula are not necessarily restricted to vector fields that satisfy the topological constraints of kinematic perfection, $dx^k - V^k dt = 0$. A discussion of topological fluctuations, where $dx^k - V^k dt = \Delta^k \neq 0$, and an example fluctuation process is described in Section 2.6.

In the next section it will be demonstrated that evolution in the direction of the Topological Torsion vector (or Current), $\mathbf{T}_4$, defined from the components of the 3-form of topological torsion,

$$i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA, \qquad (4.56)$$

induces a process which satisfies the equations of a conformal evolutionary process

$$L_{(\mathbf{T}_4)}A = \sigma A \quad \textit{and} \quad i(\mathbf{T}_4)A = 0, \;\; \sigma \neq 0, \qquad (4.57)$$

such that

$$L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = Q\hat{}dQ = \sigma^2 A\hat{}dA \neq 0. \qquad (4.58)$$

**Conclusion** *Evolution in the direction of the Topological Torsion vector, $T_4$, relative to a physical system encoded by the 1-form $A$, is thermodynamically irreversible.*

A crucial idea is to recognize that irreversible processes must be on domains of Pfaff topological dimension which support Topological Torsion, $A\hat{}dA \neq 0$, with its attendant properties of non uniqueness, envelopes, regressions, and projectivized tangent bundles Such domains are of Pfaff dimension 3 or greater. Moreover, as described below, it would appear that thermodynamic irreversibility must support a non zero Topological Parity 4-form, $dA\hat{}dA \neq 0$. Such domains are of Pfaff dimension 4 or greater.

### *4.9.3 Adiabatic Processes - Reversible and Irreversible*

The topological formulation of thermodynamics in terms of exterior differential forms permits a precise definition to be made for both reversible and irreversible adiabatic processes in terms of the topological properties of $Q$. On a geometrical space of N dimensions, a 1-form, $Q$, will admit N-1 associated vector fields, $V_{Associated}$, such that $i(V_{Associated})Q = 0$. Processes defined by associated vector fields, $V_{Associated}$, relative to $Q$ are defined as (locally) adiabatic processes (or sometimes as null vectors), $V_{adiabatic}$ [12].

$$\text{Locally Adiabatic Processes:} \quad i(V_{adiabatic})Q = 0. \tag{4.59}$$

The N-1 null vectors will form a distribution of adiabatic processes orthogonal to the 1-form $Q$. The distribution of adiabatic processes will not form a smooth hypersurface, unless the Pfaff dimension of $Q$ is 2 or less. In other words the null curves (adiabats) form a smooth hypersurface only in the equilibrium or isolated state. Note that all adiabatic processes are defined by vector direction fields, to within an arbitrary factor, $\beta(x, y, z, t)$. That is, if $i(V_A)Q = 0$, then it is also true that $i(\beta V_A)Q = 0$. The adiabatic direction fields and the 1-form of Action can be used to construct an interesting basis frame related to projective connections. This possibility will be discussed in section 5.11.

The differences between the inexact 1-forms of Work and Heat become obvious in terms of the topological format. Both 1-forms, $W$ and $Q$, depend on the process, $\mathbf{V}_4$, and on the physical system, $A$. However, Work is always transversal to the process, but Heat is not - unless the process is adiabatic:

$$\textbf{Work is transversal :} \quad i(\mathbf{V}_4)W = i(\mathbf{V}_4)i(\mathbf{V}_4)dA = 0 \tag{4.60}$$

$$\textbf{Heat is NOT transversal :} \quad i(\mathbf{V}_4)Q = i(\mathbf{V}_4)dU \nRightarrow 0, \tag{4.61}$$

$$\text{unless} \ : \ \text{the process is adiabatic} \tag{4.62}$$

It is this fundamental difference between Heat, $Q$, and Work, $W$, that lead to the Carnot-like statements that it is possible to convert work into heat with 100% efficiency, but it is not possible to convert heat into work with 100% efficiency.

Adiabatic direction fields, so defined as null curves of $Q$, do not imply that the Pfaff dimension of $Q$ must be 2. That is, it is not obvious that $Q$ can be written in the form, $Q = TdS$, as is possible on the manifold of equilibrium or isolated states. From the Cartan formulation it is apparent that if $Q$ is not zero, then

$$L_{(\mathbf{V}_A)}A = Q \neq 0, \tag{4.63}$$

$$i(\mathbf{V}_A)L_{(\mathbf{V}_A)}A = i(\mathbf{V}_A)i(\mathbf{V}_A)dA + i(\mathbf{V}_A)d(i(\mathbf{V}_A)A) \tag{4.64}$$

$$= 0(transversality) + i(\mathbf{V}_A)d(i(\mathbf{V}_A)A) = i(\mathbf{V}_A)Q$$

The necessary condition for a process to be adiabatic is given by the statement that the process is an "associated" vector relative to the exact exterior differential of the internal energy.

$$\text{An adiabatic process requires } i(\mathbf{V}_A)Q = i(\mathbf{V}_A)d(i(\mathbf{V}_A)A)) \Rightarrow 0, \tag{4.65}$$

$$Q \neq 0 \tag{4.66}$$

$$\text{with a necessary condition given by} : i(\mathbf{V}_A)dU \Rightarrow 0, \tag{4.67}$$

$$\text{and a sufficient condition given by} : d(i(\mathbf{V}_A)A) \Rightarrow 0. \tag{4.68}$$

Note that the Topological Torsion vector is an associated vector relative to the Action 1-form, $A$, and therefore defines a locally adiabatic (but irreversible) process on domains of Pfaff topological dimension 4.

If the heat 1-form is zero, then the process is a reversible adiabatic process of a special type. A reversible process is defined such that the Pfaff dimension of $Q$ is less than 3; or, $Q\hat{}dQ = 0$. Hence $i(\mathbf{V}_A)(Q\hat{}dQ) = 0$ for reversible processes. However,

$$i(\mathbf{V}_A)(Q\hat{}dQ) = (i(\mathbf{V}_A)Q)\hat{}dQ - Q\hat{}i(\mathbf{V}_A)dQ, \tag{4.69}$$

which permits reversible and irreversible adiabatic processes to be distinguished ‖ when $Q \neq 0$:

$$\textbf{Reversible Adiabatic Process} = -Q\hat{}i(\mathbf{V}_A)dQ \Rightarrow 0, \tag{4.70}$$

$$i(\mathbf{V}_A)Q \Rightarrow 0, \tag{4.71}$$

$$\textbf{Irreversible Adiabatic Process} = -Q\hat{}i(\mathbf{V}_A)dQ \neq 0, \tag{4.72}$$

$$i(\mathbf{V}_A)Q \Rightarrow 0. \tag{4.73}$$

‖It is apparent that i(V)Q = 0 defines an adiabatic process, but not necessarily a reversible adiabatic process. This topological point clears up certain misconceptions that appear in the literature.

It is certainly true that if $L_{(\mathbf{V})}A = Q = 0$, *identically*, then all such processes are adiabatic, and reversible. (In the next section, it will be demonstrated how these thermodynamic ideas can be associated with the tensor processes of covariant differentiation and parallel transport.) In such special adiabatic cases, the Cartan formalism implies that $W+dU = 0$. Such systems are elements of the Hamiltonian-Bernoulli class of processes, where $W = -dB$.

*4.9.4 Processes classified by connected topological constraints on the Work 1-form.*

Cartan has shown that all Hamiltonian processes (systems with a generator of ordinary differential equations), $\rho\mathbf{V}_H$, satisfy the following equations of topological constraint on the work 1-form, $W$ :

$$\begin{aligned} & \textbf{A Hamiltonian process } \mathbf{V}_H \text{ is either } \mathbf{V}_E \text{ or } \mathbf{V}_B \\ & \textbf{Extremal Hamiltonian } \mathbf{V}_E \\ W_E &= i(\rho\mathbf{V}_E)dA = 0 \qquad \text{Pfaff dimension of W} = 0 \qquad (4.74) \\ & \textbf{Bernoulli-Casimir Hamiltonian } \mathbf{V}_B \\ W_B &= i(\rho\mathbf{V}_B)dA = -dB \quad \text{Pfaff dimension of W} = 1 \qquad (4.75) \end{aligned}$$

More details about Cartan's development of Hamiltonian systems appears in section 3.5. A special case occurs when the Bernoulli function is equal to the negative of the internal energy, for then the heat 1-form produced by this special Hamiltonian process vanishes.

For Helmholtz processes (which are not strictly Hamiltonian) the situation is a bit more intricate, but in all cases the Pfaff dimension of the Work 1-form is at most 1. Hamiltonian processes are subsets of Helmholtz processes.

$$\begin{aligned} & \textbf{Helmholtz (Symplectic) Process } \mathbf{V}_S \\ W_S &= i(\rho\mathbf{V}_S)dA = -dB + \gamma \text{ Pfaff dimension of W} = 1 \quad (4.76) \\ dW_S &= 0 \text{ as } \gamma \text{ is closed but not exact.} \quad (4.77) \\ & \rho\mathbf{V}_S \text{ is Symplectic when} \\ dA\hat{\ }dA &\neq 0, \;\; W_S \neq 0, \;\; dW_S = 0. \quad (4.78) \end{aligned}$$

Helmholtz-Symplectic processessatisfy the following equation which is known as the Helmholtz conservation of vorticity theorem:

$$\textbf{Helmholtz} \quad : \quad \textbf{Conservation of Vorticity} \qquad (4.79)$$

$$L_{(\rho\mathbf{V}_S)}dA = dW_S + ddU = 0 + 0 = Q \Rightarrow 0. \qquad (4.80)$$

However, the closed but not exact component of work can have finite period integrals, so the evolutionary Helmholtz process can involve changing topology. The closed integrals of Action are not invariant with respect to $\rho \mathbf{V}_S$ unless $\gamma = 0$.

$$L_{(\rho \mathbf{V}_S)} \int_{z1d} A = \int_{z1d} \gamma = \int_{z1d} Q \neq 0 \tag{4.81}$$

The Helmholtz class of processes 4.81 can be split into two types:

Type $\text{H}_A$. Those processes for which the connectivity of the domain of support for the 1-form $A$ is invariant.

$$\text{Helmholtz type A}: \ L_{(\rho \mathbf{V})} \int_{z1} A \Rightarrow 0, \text{ any } \ \rho \neq 0, \int_{z1} W = \int_{z1} Q = 0. \tag{4.82}$$

Type $\text{H}_B$. Those processes for which the connectivity of the domain of support for the 1-form $A$ can change (the number of holes and handles can change),

$$\text{Helmholtz type B}: L_{(\rho \mathbf{V})} \int_{z1} A \neq 0, \ \text{ any } \rho \neq 0, \int_{z1} W = \int_{z1} Q \neq 0. \tag{4.83}$$

Cartan proved [38] that if the 1-form of Action is taken to be of the classic "Hamiltonian" format,

$$A = p_k dq^k - H(p_k, q^k, t)dt \tag{4.84}$$

on a 2n+1 dimensional domain of variables $\{p_k, q^k, t\}$, there exists a *unique* extremal vector field, $\rho V_E$, that satisfies the conditions of Helmholtz type A processes. The closed but not exact forms, $\gamma$, introduce non uniqueness into the definition of the work 1-form for Helmholtz type B processes. As $dQ = d(-dB + \gamma + dU) = 0$ for all three processes defined above, all three processes are thermodynamically reversible (see equation ( 4.49)).

**Conclusion** *Helmholtz Type B processes demonstrate that topological change is necessary but not sufficient to produce thermodynamic irreversibility.*

### *4.9.5 Planck's Harmonic Oscillator and Type B processes - How does energy get quantized ?*

Consider a symplectic Harmonic Oscillator system with a Lagrange function

$$\begin{aligned}\text{Lagrange function, } L(t,x,v) &= -1/2kx^2 + 1/2mv^2 + m_0c^2, &(4.85)\\ \text{and a 1-form of Action, } A &= pdx - (pv - L(t,x,v)dt, &(4.86)\\ \text{where } \hbar k &\doteq p - \partial L/\partial v = p - mv \neq 0. &(4.87)\end{aligned}$$

Then search for evolutionary vector fields such that the symplectic non zero virtual work is of the form:

$$\begin{aligned}W &= i(\mathbf{W})dA\\ &= [-(\hbar k)(dv - adt) + F^{diss}(dx - vdt)]\\ &= (F^{diss})dx - (\hbar k)dv + \{(\hbar k)a - F^{diss}v)dt. \qquad (4.88)\end{aligned}$$

Consider those cases where

$$\begin{aligned}F^{diss}v &= \beta\Gamma\omega v^2 &(4.89)\\ \hbar ka &= \beta\Gamma\omega xa &(4.90)\end{aligned}$$

constrained to yield the Virial Equation

$$\{(\hbar k)a - F^{diss}v) \Rightarrow \beta\Gamma\omega(xa - v^2) \Rightarrow 0. \qquad (4.91)$$

The Work 1-form then becomes

$$W = \Gamma\beta\{vd(\omega x) - (\omega x)dv\}, \qquad (4.92)$$

and if $\beta$ is chosen to be a polynomial distribution of Holder norms, where each term is of the form

$$\beta(p) = 1/\{(\omega x)^p \pm (v)^p\}^{2/p}, \qquad (4.93)$$

then each term contributes an integer to the integral

$$\oint W = \Gamma 2\pi = \sum(\text{integers}). \qquad (4.94)$$

In other words, the Virial constraint causes the 1-form of work to be of Pfaff dimension 1, $(dW = 0)$, but the 1-form of Work, $W$, is closed, but not exact. An "Open Question" remains: Does a Planck Distribution have a relationship to the polynomial of Holder norms?

Such processes on a thermodynamic system are examples of Helmholtz type B processes on symplectic manifolds. Topological fluctuations (see section 2.6) in both kinematic position and velocity are permitted, but are tamed by the constraint of the Virial condition to yield energy quantization.

> **Conclusion** *The radiation pressure (fluctuation in* $(dx - vdt)$*) or temperature (fluctuation in* $(dv - adt)$*) prevents change in the orbit. Angular momentum is constant but interaction with the environment gives a Bohr-like picture. It would appear that the application of the Virial theorem can have both statistical and topological significance. From a statistical average point of view, the Virial theorem leads to Boyle's ideal gas Law,* $PV = nRT$*. From a topological point of view the Virial theorem appears to be related to the discrete oscillation frequencies associated with quantum mechanics.*

#### *4.9.6 Locally Adiabatic Processes*

Each of the reversible processes must satisfy an additional topological constraint if the process is to be locally adiabatic:

**Locally Adiabatic Processes**

$$\text{Adiabatic process } i(\mathbf{V}_A)Q = i(\mathbf{V}_A)d(i(\mathbf{V}_A)A)) \Rightarrow 0, \quad Q \neq 0 \tag{4.95}$$

$$\text{with a sufficient condition} = i(\mathbf{V}_A)A \Rightarrow 0. \tag{4.96}$$

If $-dB = 0$, then $\rho\mathbf{V}_E$ is a characteristic process relative to the 2-form $F$. If the work 1-form is of Pfaff topological dimension 0, then the process is an extremal process relative to $A$ (see equation 4.47).

Extremal processes cannot exist on a non singular symplectic domain, because a non degenerate anti-symmetric matrix (the coefficients of the 2-form $dA$) does not have null eigenvectors on space of even dimensions . Although unique extremal stationary states do not exist on the domain of Pfaff topological dimension 4, there can exist evolutionary invariant Bernoulli-Casimir functions, $B$, that generate non extremal, "stationary"states. Such Bernoulli processes can correspond to energy dissipative Helmholtz processes, but they, as well as all Helmholtz processes, are reversible in the thermodynamic sense described in section 3.2. The mechanical energy need not be constant, but the Bernoulli-Casimir function(s), $B$, are evolutionary invariant(s), and may be used to describe non unique stationary state(s).

The equations, above, that define several familiar categories of processes, are in effect constraints the Work 1-form, $W$, generated by a process describing the topological evolution of any physical system represented by an Action 1-form, $A$. The Pfaff dimension of the 1-form of virtual work, $W = i(\mathbf{V})dA$ is 1 or less for all three sub categories of Helmholtz processes. The Extremal constraint of equation (4.74) can be used to generate the Euler equations of hydrodynamics for a incompressible fluid. The Bernoulli-Casimir constraint of equation (4.75) can be used to generate the equations for a barotropic compressible fluid. The Helmholtz constraint of equation (4.77) can be used to generate the equations for a Stokes flow. All such processes are thermodynamically reversible as $dQ = 0$. None of these constraints on the Work 1-form, $W$, above will generate the Navier-Stokes equations, which require that the topological dimension of the 1-form of virtual work must be greater than 2.

Note that for a given 1-form of Action, $A$, it is possible to construct a matrix of N-1 null (associated) vectors, and then to compute the adjoint matrix of cofactors transposed to create the unique direction field (to within a factor), $\mathbf{V}_{NullAdjoint}$. Evolution in the direction of $\mathbf{V}_{NullAdjoint}$ does not represent an adiabatic process path, as $i(\mathbf{V}_{NullAdjoint})A \neq 0$. However, for a given $A$, the N-1 null (associated) vectors represent locally adiabatic processes, but they need not span a smooth hypersurface whose surface normal is proportional to a gradient field. In fact, the components of the 1-form of Action, $A$, may be viewed as the normal vector to an implicit hypersurface, but the implicit hypersurface is not necessarily defined as the zero set of some smooth function.

### *4.9.7 Reversible processes when the Pfaff topological dimension of Work is 2 or 3*

Before studying irreversible processes, it is of some importance to study those reversible processes for which the Pfaff dimension is 2 or 3. In the process examples above, the work 1-form, $W$, was of Pfaff dimension 1 at most. As such, the Helmholtz conservation of vorticity theorem is valid, and the differential 1-form of heat is closed, $dQ = 0$. It follows that all such processes are thermodynamically reversible as $Q\hat{}dQ = 0$. However, there are processes where the work 1-form $W$ is of Pfaff dimension >1, and yet the process involved is reversible. First consider Stokes processes where the Pfaff dimension of $W$ is 2:

$$\textbf{Stokes Processes} \quad : \quad \text{If } W = -\beta dU = d(\beta U) + U d\beta, \tag{4.97}$$
$$dW = -d\beta\hat{\ } dU \tag{4.98}$$
$$Q = (1-\beta)dU, \tag{4.99}$$
$$dQ = -d\beta dU \tag{4.100}$$
$$Q\hat{\ } dQ = -(1-\beta)dU\hat{\ } d\beta\hat{\ } dU \Rightarrow 0 \tag{4.101}$$

Although the Pfaff topological dimension of the work 1-form is 2, as $Q\hat{\ } dQ = 0$, the Stokes process is a reversible process.

Next consider Chaotic reversible processes where the work 1-form is of Pfaff dimension 3. The topology induced by the work 1-form is a disconnected topology. The functions $\phi$ and $\chi$ are completely arbitrary in this example, and can be associated with the classical thermodynamic potentials. The contact structure ( as the Pfaff topological dimension of Work, $W = 3$) can be of two types: Tight and Overtwisted: Tight contact structures have a global Pfaff dimension equal to 3, Overtwisted contact structures also have a 3-form which is not zero, except at certain singular subsets. The 3-form is not global.

**Tight Contact Structures**

$$\text{If } W = \phi d\chi - dU = d(\phi\chi - U) - \chi d\phi, \tag{4.102}$$
$$dW = d\phi\hat{\ } d\chi \tag{4.103}$$
$$Q = (W + dU) \tag{4.104}$$
$$dQ = dW \tag{4.105}$$
$$Q\hat{\ } dQ = (W + dU)\hat{\ } dW \tag{4.106}$$
$$= -dU\hat{\ } dW + dU\hat{\ } dW \Rightarrow 0 \tag{4.107}$$

As $Q\hat{\ } dQ \Rightarrow 0$, these specialized processes which induce a work 1-form of Pfaff topological dimension 3 are thermodynamically reversible. The Work 1-form generates a contact 3 manifold which has no limit cycles [61]. It will be shown below (see section 2.6.3) how such processes are related to the classical thermodynamic potentials, for specific choices of the function $(\phi\chi - U)$.

Neither of these last 2 processes conserve vorticity (think angular momentum). Yet they are candidates for investigating reconnection processes [85].

Of particular interest are those processes for which the work 1-form generates an "Over-twisted Contact structure". Such structures are important for they are the domain of limit cycles. As an example define the Holder function as a quadratic form in terms of two independent functions, $\phi$ and $\chi$ , as:

**Definition** *The "Holder" variable is defined as* $h^2 = \phi^2 \pm \chi^2$

A constant value for the square Holder norm is elliptic or hyperbolic depending upon the $\pm$ sign. Next define the closed 1-form of Pfaff dimension 1, as

**Definition** *The closed but not exact 1-form, $\gamma$, is defined as*

$$\gamma = (\phi d\chi - \chi d\phi)/(\phi^2 \pm \chi^2) = (\phi d\chi - \chi d\phi)/h^2, \quad (4.108)$$

$$d\gamma \Rightarrow 0. \quad (4.109)$$

The closed form $\gamma$ plays the role of a "differential angle variable $\delta\theta$" in the elliptic case.

Now study those processes where the work 1-form is of Pfaff dimension 3, but not globally.

**Over-twisted Contact Structures**

**( Limit cycles )**

$$\text{If } W = f(h)\gamma - dU, \quad (4.110)$$

$$dW = \partial f/\partial h dh\hat{}\gamma \quad (4.111)$$

$$W\hat{}dW = \{-\partial f/\partial h\ \}dU\hat{}dh\hat{}\gamma \quad (4.112)$$

$$Q = (W + dU), dQ = dW, \quad (4.113)$$

$$Q\hat{}dQ = -dU\hat{}dW + dU\hat{}dW \Rightarrow 0 \quad (4.114)$$

Although the Work 3-form $W\hat{}dW$ is not zero almost everywhere, the heat 3-form $Q\hat{}dQ = 0$ is zero globally, Hence the process is thermodynamically reversible. However, the 3-form volume element created by the work 1-form is not global and will admit defect structures. In the example above, the work $3-$form, $W\hat{}dW$, considered as a 3D - volume element, has singularities which occur at the zeros of the function $-\partial f/\partial h$ . If, for example,

$$f(h) = (b + h - h^3/3a^2), \quad (4.115)$$

then the circle, $-\partial f/\partial h = 0$, defines a limit cycle in the elliptic case, in the two dimensional plane defined by $\phi$ and $\chi$ :

$$-\partial f/\partial h = -\left(1 - h^2/a^2\right) \Rightarrow 0 \quad (4.116)$$

$$h^2 = \phi^2 + \chi^2 = a^2. \quad (4.117)$$

The limit cycle is stabile (attracting) if the volume element has a negative orientation (contracting), and is unstable otherwise. To cement the ideas, rewrite the work 1-form in terms of a more suggestive set of symbols, and observe that the rotational term has the format of the component of angular momentum orthogonal to the plane of rotation.

$$\text{If } W = i(\rho \mathbf{V}_4)dA = m\Gamma(h)(xdy - ydx) - dU \quad (4.118)$$

$$\Rightarrow \Gamma(h)\{m(xV^y - yV^x)\}dt - dU, \quad (4.119)$$

$$= \Gamma(h)\mathbf{L}_z dt - dU. \quad (4.120)$$

As $d\gamma = 0$ except at the fixed point of the "rotation", the Pfaff dimension of $W$ has evolved from Pfaff dimension 3 to Pfaff dimension 1, as $\partial\Gamma/\partial h \Rightarrow 0$. In the Pfaff dimension 1 state, Helmholtz theorem becomes valid and "vorticity" is preserved. In the Pfaff dimension 3 mode, the process does not conserve vorticity. When the system decays (or is attracted) to the Pfaff dimension 1 state, the subsequent work done by a cyclic process is not necessarily zero. The closed but not exact 1-form $\gamma$ can contribute to a period integral. Upon reflection, what has been described is the approach (Pfaff dimension 3) to a limit cycle (Pfaff dimension 1). The entire process has been done reversibly. Other forms of both the tight and the overtwisted contact structures defined by the work 1-form, can occur and such C2 processes can be thermodynamically irreversible. However, it will be demonstrated below that sequential C1 processes exist for all contact structures that are thermodynamically reversible.

## 4.10 A Physical System with Topological Torsion

For maximal, non equilibrium, turbulent systems in space-time, the maximal element in the Pfaff sequence generated by $A, W$, or $Q$, is a 4-form. On the geometric space of 4 independent variables, every 4-form is globally closed, in the sense that its exterior differential vanishes everywhere. It follows that every 4-form is exact and can be generated by the exterior differential of a 3-form. The exterior differential of the 3-form is related to the concept of a divergence of a contravariant vector field. A large fraction of the development in this monograph will be devoted to the study of such 3-forms, and their kernels, for it is 3-forms that form indicators of non equilibrium systems and processes. It is a remarkable fact that all 3-forms (in general, N-1 forms) admit integrating denominators, such that the exterior differential of a rescaled 3-form is zero almost everywhere. Space time points upon which the integrating denominator has a zero value produce singularities defined as topological defect structures.

When the Action for a physical system is of Pfaff dimension 4, there exists a unique direction field, $\mathbf{T}_4$, defined as the Topological Torsion 4-vector, that can be evaluated *entirely* in terms of those component functions of the 1-form of Action which define the physical system. To within a factor, this direction field** has the four components of the 3-form $A\hat{}dA$, with the following properties:

**Properties of the Topological Torsion vector** $\mathrm{T}_4$

$$i(\mathbf{T}_4)\Omega_4 = A\hat{}dA \tag{4.121}$$

$$W = i(\mathbf{T}_4)dA = \sigma\ A, \tag{4.122}$$

$$U = i(\mathbf{T}_4)A = 0, \tag{4.123}$$

$$L_{(\mathbf{T}_4)}A = \sigma\ A, \tag{4.124}$$

$$Q\hat{}dQ = L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = \sigma^2 A\hat{}dA \neq 0 \tag{4.125}$$

$$dA\hat{}dA = (2!)\ \sigma\ \Omega_4. \tag{4.126}$$

Note that a $\mathbf{T}_4$ process is locally adiabatic.

Hence, by equation (4.125 ) evolution in the direction of $\mathbf{T}_4$ is thermodynamically irreversible, when $\sigma \neq 0$ and $A$ is of Pfaff topological dimension 4. The kernel of this vector field is defined as the zero set under the mapping induced by exterior differentiation. In engineering language, the kernel of this vector field are those point sets upon which the divergence of the vector field vanishes. The Pfaff topological dimension of the Action 1-form is 3 in the defect regions defined by the kernel of $\mathbf{T}_4$. The coefficient $\sigma$ can be interpreted as a measure of space-time volumetric expansion or contraction. It follows that both expansion and contraction processes (of space-time) are related to irreversible processes. It is here that contact is

**A direction field is defined by the components of a vector field which establish the "line of action" of the vector in a projective sense. An arbitrary factor times the direction field defines the same projective line of action, just reparameterized. In metric based situations, the arbitrary factor can be interpreted as a renormalization factor.

made with the phenomenological concept of "bulk" viscosity $= (2!)\sigma$. (For symplectic systems of higher Pfaff dimension $m = 2n + 2 \geq 4$, the numeric factor becomes $(m/2)!$.) It is important to note that the concept of an irreversible process depends on the square of the coefficient, $\sigma$. It follows that both expansion and contraction processes (of space-time) are related to irreversible processes. It is tempting to identify $\sigma^2$ with the concept of entropy production.

The Topological Torsion vector vanishes when the Pfaff topological dimension of A is 2 or less. Note that the Frenet-Serret geometric torsion of a space curve vanishes when the geometric dimension is 2 or less. It is this analog dependence on dimension 3 or more that led to the name "Topological Torsion" for the 3-form $A\hat{}dA$. Solution uniqueness is lost when the Topological Torsion vector is not zero. In 4D, the three form $A\hat{}(dA)$ has been defined as the Topological Torsion 3-form. The Torsion current depends only on the system (the Action) and not upon a process. The divergence of this Torsion current is proportional to the measure of the 4D volume, that defines the symplectic space, and cannot be zero on the symplectic domain. The components of the Topological Torsion vector $\mathbf{T}_4$ generate what is called the "subsidiary Pfaffian system" by Forsyth [68].

For purposes of more rapid comprehension, consider a 1-form of Action, $A$, with an exterior differential, $dA$, and a notation that admits an electromagnetic interpretation ($\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi$, and $\mathbf{B} = \nabla \times \mathbf{A}$)[††]. The explicit format of the Electromagnetic Topological Torsion 4 vector, $\mathbf{T}_4$ becomes:

$$\mathbf{T}_4 = -[\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi,\ \mathbf{A} \circ \mathbf{B}] \tag{4.127}$$

$$A\hat{}dA = i(\mathbf{T}_4)\Omega_4 \tag{4.128}$$

$$= T_4^x dy\hat{}dz\hat{}dt - T_4^y dx\hat{}dz\hat{}dt + T_4^z dx\hat{}dy\hat{}dt - T_4^t dx\hat{}dy\hat{}dz, \tag{4.129}$$

$$dA\hat{}dA = 2(\mathbf{E} \circ \mathbf{B})\ \Omega_4 = K\Omega_4 \tag{4.130}$$

$$= \{\partial T_4^x/\partial x + \partial T_4^y/\partial y + \partial T_4^z/\partial z + \partial T_4^t/\partial t\}\ \Omega_4. \tag{4.131}$$

When the divergence of the topological torsion vector is not zero, $\sigma = (\mathbf{E} \circ \mathbf{B}) \neq 0$, and $A$ is of Pfaff dimension 4, $W$ is of Pfaff dimension 4, and $Q$ is of Pfaff dimension 4. The process generated by $\mathbf{T}_4$ is thermodynamically

[††]The bold letter **A** represents the first 3 components of the 4 vector of potentials, with the order in agreement with the ordering of the independent variables. The letter A represents the 1-form of Action.

irreversible, as

$$Q\hat{}dQ = L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = \sigma^2 A\hat{}dA \neq 0. \tag{4.132}$$

The evolution of the volume element relative to the irreversible process $\mathbf{T}_4$ is given by the expression,

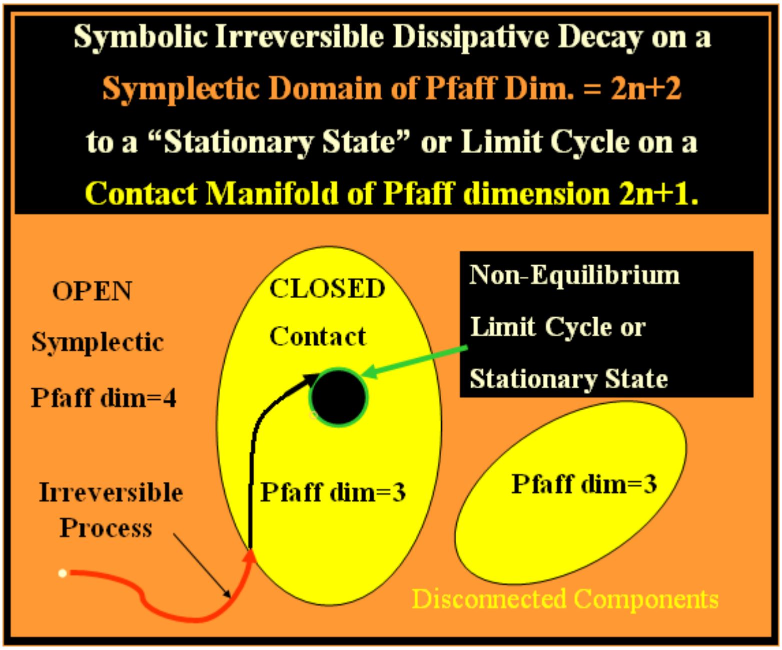
$$L_{(\mathbf{T}_4)}\Omega_4 = i(\mathbf{T}_4)d\Omega_4 + d(i(\mathbf{T}_4)\Omega_4) \tag{4.133}$$

$$= 0 + d(A\hat{}dA) = 2(\mathbf{E} \circ \mathbf{B})\ \Omega_4. \tag{4.134}$$

Hence, the differential volume element $\Omega_4$ is expanding or contracting depending on the sign and magnitude of $\mathbf{E} \circ \mathbf{B}$, a useful fact when topological thermodynamics is applied to cosmology. The irreversible dissipation induced by a $\mathbf{T}_4$ process can be compared to a bulk viscosity coefficient. A cosmology on 4D can have an expanding volume element, $\Omega_4$, but with embedded 3D defect structures (the galaxies) which are not "expanding".

If $A$ is (or becomes) of Pfaff dimension 3, then $dA\hat{}dA \Rightarrow 0$ which implies that $\sigma^2 \Rightarrow 0$, but $A\hat{}dA \neq 0$. The differential geometric volume element $\Omega_4$ is subsequently an evolutionary invariant, and evolution in the direction of the topological torsion vector is thermodynamically reversible. The physical system is not in equilibrium, but the divergence free $\mathbf{T}_4$ evolutionary process forces the Pfaff dimension of $W$ to be zero, and the Pfaff dimension of $Q$ to be at most 1. Indeed, a divergence free $\mathbf{T}_4$ evolutionary process has a Hamiltonian representation, and belongs to the characteristic class of vector fields.

In the domain of Pfaff dimension 3 for the Action, $A$, the subsequent continuous evolution of the system, $A$, relative to the process $\mathbf{T}_4$, proceeds in an energy conserving manner, representing a "stationary" or "excited" state far from equilibrium. These excited states can be interpreted as the evolutionary topological defects in the Turbulent dissipative system of Pfaff dimension 4. The Topological Torsion vector becomes an adiabatic, extremal, characteristic direction field in the space of geometric dimension 4, but where the Pfaff dimension of the physical system, $A$, is of Pfaff topological dimension 3.




On a geometric domain of 4 dimensions, assume that the evolutionary process generated by $\mathbf{T}_4$ starts from an initial condition (or state) where the Pfaff topological dimension of $A$ is also 4. Depending on the sign of the divergence of $\mathbf{T}_4$, the process follows an irreversible path for which the divergence represents an expansion or a contraction. If the irreversible evolutionary path is attracted to a region (or state) where the Pfaff topological dimension of the 1-form of Action is 3, then $\mathbf{E} \circ \mathbf{B}$ becomes (or has decayed to) zero. The zero set of the function $\mathbf{E} \circ \mathbf{B}$ defines a hypersurface in the 4 dimensional space. If the process remains trapped on this hypersurface of Pfaff dimension 3, $\mathbf{E} \circ \mathbf{B}$ remains zero, and the $\mathbf{T}_4$ process becomes an extremal, adiabatic, characteristic direction field. Such extremal fields are such that the virtual work 1-form vanishes, $W = i(\mathbf{T}_4)dA = 0$. The direction field that represented an irreversible process, in domains where the divergence goes to zero, becomes a representation for a reversible conservative extremal Hamiltonian process. Although the extremal process is conservative in a Hamiltonian sense, the physical system can be in a "excited" state on the hypersurface that is far from equilibrium, for the Pfaff dimension of the 1-form of Action is 3, and not 2. (If the path is attracted to a region where the function $\mathbf{E} \circ \mathbf{B}$ is oscillatory, the system evolutionary path defines a limit cycle, or what has been called a "breather".)

The fundamental claim made in this monograph is that it is these topological defects that self organize from the dissipative irreversible evolution of the Turbulent State into "stationary" states far from equilibrium. These long lived stationary states form the stars and the galaxies of the cosmos at a cosmological level. They represent the long lived remnants or wakes generated from irreversible processes in a dissipative non equilibrium macroscopic turbulent fluid. On another scale, these topological defects form the excited quantum states at the microscopic level.

## 4.11 The Lie differential $L_{(V)}$ and the Covariant differential $\nabla_{(V)}$

The covariant derivative of tensor analysis, as used in General Relativity, is often defined in terms of isometric diffeomorphic processes (that preserve the differential line element) and can be used to describe rigid body motions and isometric bendings, but not deformations and shear processes associated with convective fluid flow. Another definition of the covariant derivative is based on the concept of a connection, such that the differential process acting on a tensor produces a tensor. The definition of the covariant derivative usually depends upon the additional structure (or constraint) of a metric or a connection placed on a given variety, while the Lie differential does not. As the Lie differential is not so constrained, it may be used to describe non diffeomorphic processes for which the topology changes continuously. The covariant derivative is avoided in this monograph.

Koszul (see p 262 in [82]) has given a set of axioms that can be used to define a linear affine connection and a covariant derivative. The covariant derivative axioms require that

$$\nabla_{(fV)}\omega = f\ \nabla_{(V)}\omega, \tag{4.135}$$

$$\nabla_{(V)}f\omega = (\nabla_{(V)}f\ )\omega + f\ \nabla_{(V)}\omega. \tag{4.136}$$

This axiomatic representation of a covariant derivative and an affine connection should be compared to the Lie differential,

$$L_{(fV)}A = f\ L_{(V)}A + df\ (i(V)A), \tag{4.137}$$

$$L_{(V)}fA = (L_{(V)}f\ )A + f\ L_{(V)}A. \tag{4.138}$$

Only if the last term in the expansion of the Lie differential, $df\ (i(V)A)$, is zero does the formula for the Lie differential have an equivalent representation as a covariant derivative in terms of a connection. Suppose that $i(V)A = 0$, such that the Lie differential and the covariant differential are equivalent.

$$L_{(fV)}A = f\ L_{(V)}A = f\ \nabla_{(V)}A. \tag{4.139}$$

Then it follows that

$$\begin{aligned} L_{(fV)}A &= f\ L_{(V)}A + df\ (i(V)A) && (4.140)\\ &= f\ L_{(V)}A = f\ Q. && (4.141)\\ \text{But}\ \ i(V)Q &= f\ i(V)i(V)dA \Rightarrow 0 && (4.142)\\ \text{where}\ i(V)Q &= 0 \text{ defines an adiabatic process.} && (4.143) \end{aligned}$$

> **Theorem 2.** Hence, *all covariant derivatives with respect to an affine connection have an equivalent representation as an adiabatic process!!!* Such covariant adiabatic processes need not be thermodynamically reversible.

Suppose that the adiabatic process is such that

$$L_{(V)}A = \ Q = 0. \tag{4.144}$$

Then

$$dL_{(V)}A = L_{(V)}dA = \ dQ = 0, \tag{4.145}$$

and it follows that the adiabatic process is reversible. However, the condition that $Q$ be zero is the equivalent to the condition of parallel transport:

$$L_{(V)}\omega \Rightarrow \nabla_{(V)}\omega = 0. \tag{4.146}$$

> **Theorem 3.** The remarkable conclusion is that *the concept of parallel transport in tensor analysis is - in effect - an adiabatic, reversible process*!!!

As it is a matter of experience that not all evolutionary processes are adiabatic, much less reversible, it seems sensible to conclude that theories (such as general relativity) that invoke the use of a covariant derivative, and or parallel transport, to describe evolutionary processes have allowed irreversible phenomena, in the words of Sir Arthur Eddington, "to slip through the net".

## 4.12 Topological Fluctuations.

### *4.12.1 The Cartan-Hilbert Action 1-form..*

This subsection considers those physical systems that can be described by a Lagrange function $L(\mathbf{q},\mathbf{v},t)$ and a 1-form of Action given by:

$$A = L(\mathbf{q}^k,\mathbf{v}^k,t)dt + \mathbf{p}_k \cdot (d\mathbf{q}^k - \mathbf{v}^k dt), \tag{4.147}$$

The classic Action, $L(\mathbf{q}^k,\mathbf{v}^k,t)dt$, is extended to included fluctuations in the kinematic variables. It is no longer assumed that the equation of Kinematic perfection is satisfied. Fluctuations of the topological constraint of kinematic perfection are permitted:

$$\textbf{Topological Fluctuations in position:}\ \ \Delta\mathbf{q} = (d\mathbf{q}^k - \mathbf{v}^k dt) \neq 0. \tag{4.148}$$

When dealing with fluctuations, the geometric dimension will not be constrained to 4 independent variables. At first glance it appears that the domain of definition is a 3n+1 dimensional variety of independent base variables, $\{\mathbf{q}^k,\mathbf{v}^k,t\}$. Do not assume that $\mathbf{p}$ is constrained to be a jet; e.g., $\mathbf{p}_k \neq \partial L/\partial \mathbf{v}^k$. Instead, consider $\mathbf{p}_k$ to be a (set of) Lagrange multiplier(s) to be determined later. Note that the Action 1-form has the format used in the Cartan-Hilbert invariant integral [45], except that it is not assumed that $\mathbf{p}_k$ is canonical; $\mathbf{p}_k \neq \partial L/\partial \mathbf{v}^k$ necessarily. Also, do not assume at this stage that $\mathbf{v}$ is a kinematic velocity function, such that $(d\mathbf{q}^k - \mathbf{v}^k dt) \Rightarrow 0$. The classical idea is to assert that topological fluctuations in kinematic velocity are related to pressure.

For the given Action, construct the

$$\text{Pfaff sequence}\quad \{A, dA, A\hat{}dA, dA\hat{}dA...\} \tag{4.149}$$

in order to determine the Pfaff dimension or class of the 1-form [123]. The top Pfaffian is defined as the non zero p-form of largest degree p in the sequence). The top Pfaffian for the Cartan-Hilbert Action is given by the formula

**Top Pfaffian 2n+2**

$$(dA)^{n+1} = (n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\}\hat{}\Omega_{2n+1} \tag{4.150}$$

$$\Omega_{2n+1} = dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt, \tag{4.151}$$

The formula is a bit surprising in that it indicates that the Pfaff topological dimension of the Cartan-Hilbert 1-form is 2n+2, and not the geometrical dimension $3n+1$. For n = 3 degrees of freedom, the top Pfaffian indicates that the topological of Pfaff topological dimension of the 2-form, $dA$ is $2n + 2 = 8$. The value $3n + 1 = 10$ might be expected as the 1-form was defined initially on a space of $3n + 1$ "independent" base variables. The implication is that there exists an irreducible number of independent variables equal to $2n+2 = 8$ which completely characterize the differential topology of the first order system described by the Cartan-Hilbert Action. It follows that the exact two form, $dA$, satisfies the equations

$$(dA)^{n+1} \neq 0, \; but \quad A\hat{}(dA)^{n+1} = 0. \tag{4.152}$$

The format of the top Pfaffian requires that the bracketed factor

$$\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\} \tag{4.153}$$

can be represented (to within a factor) by a perfect differential:

$$dS = (n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\} \tag{4.154}$$

The result is also true for any closed addition $\gamma$ added to $A$; e.g., the result is "gauge invariant". Addition of a closed 1-form does not change the Pfaff dimension from even to odd. On the other hand the result is not renormalizable, for multiplication of the Action 1-form by a function can change the algebraic Pfaff dimension from even to odd.

On the 2n+2 domain, the components of 2n+1 form $T = A\hat{}(dA)^n$ generate what has been defined herein as the Topological Torsion vector, to within a factor equal to the Torsion Current. The coefficients of the 2n+1 form are components of a contravariant vector density $\mathbf{T}^m$ defined as the Topological Torsion vector, the same concept as defined previously, but now extended to 2n+2 dimensions. This vector is orthogonal (transversal) to the 2n+2 components of the covector, $\mathbf{A}_m$. In other words,

$$A\hat{}T = A\hat{}(A\hat{}(dA)^n) = 0 \Rightarrow i(\mathbf{T})(A) = \sum \mathbf{T}^m \mathbf{A}_m = 0. \tag{4.155}$$

This result demonstrates that the extended Topological Torsion vector represents an adiabatic process. This topological result does not depend upon geometric ideas such as metric. In section 3.3.2 it was demonstrated, on a space of 4 independent variables, that evolution in the direction of the Topological Torsion vector is irreversible in a thermodynamic sense, subject to the symplectic condition of non zero divergence, $d(A\hat{}dA) \neq 0$. The same result holds on dimension 2n+2.

The 2n+2 symplectic domain so constructed can not be compact without boundary for it has a volume element which is exact. By Stokes theorem, if the boundary is empty, then the surface integral is zero, which would require that the volume element vanishes; but that is in contradiction to the assumption that the volume element is finite. For the 2n+2 domain to be symplectic, the top Pfaffian can never vanish. The domain is therefore orientable, but has two components, of opposite orientation. Examination of the constraint that the symplectic space be of dimension 2n+2 implies that the Lagrange multipliers, $\mathbf{p}_k$, cannot be used to define momenta in the classical "conjugate or canonical" manner. Define the non canonical components of the momentum, $\hbar k_j$, as

$$\textbf{non canonical momentum: } \hbar k_j = (p_j - \partial L/\partial v^j), \tag{4.156}$$

such that the top Pfaffian can be written as

$$(dA)^{n+1} = (n+1)!\{\Sigma_{j=1}^{n} \hbar k_j dv^j\}\hat{}\,\Omega_{2n+1}. \tag{4.157}$$

For the Cartan Hilbert Action to be of Pfaff topological dimension 2n+2, the factor $\{\Sigma_{j=1}^{n} \hbar k_j dv^j\} \neq 0$. It is important to note, however, that as $(dA)^{n+1}$ is a volume element of geometric dimension 2n+2, the 1-form $\Sigma_{j=1}^{n} \hbar k_j dv^j$ is exact (to within a factor, say $T(\mathbf{q}^k, t,\ \mathbf{p}_k, S_{\mathbf{v}})$):

$$\Sigma_{j=1}^{n} \hbar k_j dv^j = T dS_{\mathbf{v}}. \tag{4.158}$$

Tentatively, this 1-form, $dS_{\mathbf{v}}$, will be defined as the Topological Entropy production relative to fluctuations of differential position. The concept of entropy with respect to continuous topological evolution will be discussed in more detail in Section 2.7.

### *4.12.2 Thermodynamics and Topological Fluctuations of Work*

Topological fluctuations are admitted when the evolutionary vector direction fields are not singly parametrized. It is historical to consider the interpretations of equilibrium statistical fluctuations in terms of pressure and temperature. These concepts are assumed to be transported to topological fluctuations:

$$\text{Position Fluctuations (pressure)}: \quad d\mathbf{q} - \mathbf{v}dt = \Delta\mathbf{q} \neq 0 \tag{4.159}$$

$$\text{Velocity Fluctuations (temperature)}: \quad d\mathbf{v} - \mathbf{a}dt = \Delta\mathbf{v} \neq 0 \tag{4.160}$$

These "failures" of kinematic perfection undo the topological refinements imposed by a "kinematic particle" point of view, and place emphasis on the continuum methods inherent in fluids and plasmas.

For the maximal non canonical symplectic physical system of Pfaff dimension 2n+2, consider evolutionary processes to be representable by vector fields of the form $\gamma V_{3n+1} = \gamma\{\mathbf{v}, \mathbf{a}, \mathbf{f}, 1\}$, relative to the independent variables $\{\mathbf{q}, \mathbf{v}, \mathbf{p}, t\}$. Use the Cartan magic formula definition of the "virtual work" 1-form, $W$, as $W = i(\gamma V_{3n+1})dA$. The Work 1-form must vanish for the case of extremal evolution, and be non zero, but closed, for the case of symplectic evolution

First compute the 2-form, $dA$ from the Cartan-Hilbert Action:

$$dA = \{\partial L/\partial v^k - p_k\}(\Delta v^k)\hat{}dt + \{dp_k - \partial L/\partial x^k dt\}\hat{}(\Delta q^k) \tag{4.161}$$

Then compute the Work 1-form

$$W = (\gamma V_{3n+1})dA = \{\mathbf{p} - \partial L/\partial \mathbf{v}\} \bullet \Delta\mathbf{v} + \{\mathbf{f} - \partial L/\partial \mathbf{q}\} \bullet \Delta\mathbf{q} \tag{4.162}$$

Note that $\{\mathbf{p} - \partial L/\partial \mathbf{v}\}$ is the definition of the non canonical momentum, $\hbar k_j$, and $\{\mathbf{f} - \partial L/\partial \mathbf{q}\}$ represents those components of the force that are not conservative. When the fluctuations in velocity are zero (temperature) and the fluctuations in position are zero (pressure), then the work 1-form will vanish, and the process and physical system admits an extremal Hamiltonian representation. On the other hand if the fluctuations in velocity are not zero and the fluctuations in position are not zero, then the Work 1-form vanishes only if the momenta (the Lagrange multipliers, $\mathbf{p}$, are canonically defined ($\{\mathbf{p} - \partial L/\partial \mathbf{v}\} \Rightarrow 0$) and the Newtonian force is a gradient, $\{\mathbf{f} - \partial L/\partial \mathbf{x}\} \Rightarrow 0$. These topological constraints are ubiquitously assumed in classical conservative Hamiltonian mechanics.

When all topological fluctuations vanish, then the Pfaff dimension of the work 1-form is also zero. This is a sufficient but not necessary condition for equilibrium.[‡‡] It is possible that when the momenta are canonical, and the force is conservative, the equilibrium state can admit fluctuations, and yet the Work 1-form vanishes and the Heat 1-form is exact. This result can be used as starting point for a statistical analysis of the equilibrium state (statistical methods are more or less ignored in this monograph).

### Fluctuations in Pfaff topological dimension 2n+2 and 2n+1

When the 2-form $dA$ is non zero, all processes acting on the Cartan Hilbert Action, generate a work 1-form of the form given in equation (4.162). The

[‡‡]Inanimate is perhaps a better description of the state with zero fluctuations.

maximum topological dimension for the Cartan-Hilbert Action is 2n+2. Suppose that the 2-form $dA$ is constructed in terms of these "2n+2 topological coordinates". The 2-form $dA$ is said to be non degenerate, or of maximal rank, on the $2n+2$ dimensional space in regions where the antisymmetric matrix representing $dA$ has no zero eigenvalues. (Recall that closed non degenerate 2-forms define a symplectic structure [123]). However, there may exist singularities in the space of topological coordinates 2n+2 where the 2-form $dA$ becomes "singular". In such $2n+2$ regions the 2-form $dA$ on the $2n+2$ space becomes degenerate and admits zero eigenvalues. Such regions, where $dA$ is of Pfaff dimension $2n+1$, can be considered to be topological defects or subspaces in the 2n+2 topological domain. In such subspaces, the 2-form $dA$, expressed in terms of the 2n+2 topological coordinates, admits two null eigenvectors. As the eigen values of an anti-symmetric matrix of $(2n+2)\times(2n+2)$ functional elements come in pairs, vectors representing topological defects of the symplectic domain are not unique, a well known result of the calculus of variations having envelope solutions (see section 8.7). One of these null eigen vectors (of $2n+2$ components) is the unique Hamiltonian-extremal field, and the other is the topological torsion vector (of $2n+2$ components and zero divergence), which is reduced to a Characteristic vector relative to the Action in the subspace of topological defects. The Characteristic vector is equivalent to the Topological Torsion vector, only if the divergence of the Topological Torsion vector is zero. The second null solution vector (which is adiabatic in a thermodynamic sense) can be related to the Hamilton Jacobi theory in classical mechanics. Processes defined by the extremal field or the characteristic field (degenerate topological torsion vector) are thermodynamically *reversible.* In contrast, the process generated by the topological torsion vector with non zero divergence is thermodynamically irreversible.

These facts can now be combined with the expression for the work 1-form given in equation (4.162). In the regions where $dA$ is non degenerate, the Work cannot vanish (as this would imply a null eigenvector). It follows that the following 4 situations are NOT allowed when $dA$ is of maximal rank.

**Case 1. Canonical momentum and gradient forces**

$$\{\mathbf{p}-\partial L/\partial\mathbf{v}\}=0 \; and \;\; \{\mathbf{f}-\partial L/\partial\mathbf{q}\}=0. \tag{4.163}$$

**Case 2. Canonical momentum and zero kinematic fluctuations in position.**

$$\{\mathbf{p}-\partial L/\partial\mathbf{v}\}=0 \; and \;\; \Delta\mathbf{q}=0. \tag{4.164}$$

**Case 3. Zero kinematic fluctuations in velocity and gradient forces**

$$\Delta\mathbf{v} = 0 \; and \; \{\mathbf{f} - \partial L/\partial\mathbf{q}\} = 0. \tag{4.165}$$

**Case 4. Zero kinematic fluctuations in velocity and Zero kinematic fluctuations in position**

$$\Delta\mathbf{v} = 0 \; and \; \Delta\mathbf{q} = 0. \tag{4.166}$$

Conversely, when $dA$ generates a contact manifold of Pfaff topological dimension 2n+1, one of the four cases above must be true. In the contact 2n+1 domain, however, there exists a unique vector field with a null eigen value, such that the virtual work 1-form indeed vanishes: $W = i(\mathbf{X})dA = 0$. This result serves as the basis of the d'Alembert principle. An elementary case is based upon the assumption that Case 4 is valid. That is, there exists a kinematic description of the process at both the first and the second order (velocities and accelerations are singly parameterized). Another case that is common is based on the assumption that the momentum is canonically defined. Then, for the Contact extremal case to exist, and as $\{\mathbf{p}-\partial L/\partial\mathbf{v}\} = 0$, it is necessary that the work 1-form reduces to vanishing expression

$$W = \{\mathbf{f} - \partial L/\partial\mathbf{q}\} \circ \Delta q \Rightarrow 0 \text{ in the extremal case.} \tag{4.167}$$

The extremal constraint is satisfied when the bracket factor vanishes, which is then the equivalent of the Lagrange-Euler equations of classical mechanics. However, the Contact constraints are also satisfied when the force is a gradient field, or there exist zero fluctuations in position, or the non zero components of the force (the otherwise dissipative components) are orthogonal to the kinematic fluctuations in position.

### Bernoulli-Hamiltonian Processes and fluctuations in Work

A Bernoulli-Hamiltonian process is not uniquely defined by the 1-form of Action representing the physical process. Recall that the extremal direction field in the domain of Pfaff dimension 2n+1 and the topological torsion direction field in the domain of Pfaff dimension 2n+2 are uniquely defined by the functional format of the 1-form of Action representing the physical system. Further recall that a Bernoulli-Hamiltonian process is defined by the Work 1-form being non zero and exact, $W = i(\mathbf{X})dA = -dB \neq 0$, where $B$ is an arbitrary function, often called a "Casimir" - or somewhat inappropriately, a "Hamiltonian". In non singular regions where the 1-form $A$ is of Pfaff dimension 2n+2, and is non degenerate, the functions $B$ are never constant and never without a gradient. Although they are not constants over the domain, these "potential" or "energy" functions $B$ are evolutionary invariants of the Bernoulli-Hamiltonian process, $\mathbf{X}$. That is, a Bernoulli function, $B$, is an invariant along a given path, but can have different values for $B$ on neighboring paths.

Most engineers and applied scientists have a greater appreciation for these functions when it is pointed out that they are equivalent to the Bernoulli invariants in hydrodynamics and the thermodynamic potentials in classical thermodynamics. The engineer would call $B$ a Bernoulli "constant", a function invariant along a streamline, but which has different values for different neighboring streamlines: $B = (P + \rho gh + \rho v^2/2)$.

To prove that the Bernoulli-Casimirs are always evolutionary invariants with respect to the vector fields, $\mathbf{X}$, construct the Lie differential of $B$ with respect to $\mathbf{X}$.

$$L_{(\mathbf{X})}B = i(\mathbf{X})dB + d(i(\mathbf{X})B) = i(\mathbf{X})i(\mathbf{X})dA + d(i(\mathbf{X})B) = 0 + 0. \quad (4.168)$$

Both the first and second terms vanish algebraically. However, for the classic "Hamiltonian" defined above in terms of the Legendre transformation, $H(t, q, v, p) = \{p_k v^k - L(t, q^k, v^k\}$, a direct computation indicates that the Hamiltonian need not be an invariant of a symplectic process - even if the Hamiltonian is explicitly time independent. For consider the evolutionary equation,

$$L_{(\mathbf{X})}H = i(\mathbf{X})dH = \{(\partial H/\partial \mathbf{q}) \bullet \mathbf{v} + (\partial H/\partial \mathbf{p}) \cdot \mathbf{f} + (\partial H/\partial \mathbf{v}) \cdot \mathbf{a} + (\partial H/\partial t)\} \quad (4.169)$$

or equivalently

$$L_{(\mathbf{X})}H = \{(\mathbf{p} - \partial L/\partial \mathbf{v}) \bullet \mathbf{a} + (\mathbf{f} - \partial L/\partial \mathbf{q}) \bullet \mathbf{v} - (\partial L/\partial t)\}. \quad (4.170)$$

For the domain of the Cartan-Hilbert Action which is of Pfaff topological dimension 2n+2, the first factor of the first term cannot vanish. The first factor of the second term, when set to zero, is equivalent to the classical Lagrange-Euler equations, and the forces are conservative gradient fields. Suppose that $(\partial L/\partial t) = -(\partial H/\partial t) = 0$, and the non conservative forces are orthogonal to the velocities, then, even in this case, if the accelerations $\mathbf{a}$ are such that $(\mathbf{p} - \partial L/\partial \mathbf{v}) \cdot \mathbf{a} \neq 0$, the "Hamiltonian energy" $H$, is not an evolutionary invariant relative to $\mathbf{X}$. Yet the Bernoulli-Casimir energies are evolutionary invariants relative to $\mathbf{X}$. A simple example of this situation is where the mechanical (Hamiltonian) energy of a system decays to perhaps some non zero value at a singular point of the 2n+2 domain, but the angular momentum stays constant during the process. Numerical simulations of such evolutionary possibilities in fluids have been studied by Carnevale [34].

**Thermodynamic Potentials and Reversible Processes as Bernoulli evolutionary invariants.**

From the topological version of the first law in terms of Cartan's magic formula, and from the concept that thermodynamic reversibility requires that $Q\hat{}dQ = 0$, it follows that for reversible processes,

$$Q\hat{}dQ = (W + dU)\hat{}dW = W\hat{}dW + dU\hat{}dW \Rightarrow 0 \tag{4.171}$$

Suppose that the Work 1-form is restricted to the format of Pfaff dimension 3

$$\begin{aligned} \text{Pfaff dimension 3,}\ W &= -dU + \phi\hat{}d\chi && (4.172)\\ &= d\{-U + \phi\chi\} - \chi d\phi, && (4.173)\\ dW &= d\phi\hat{}d\chi, && (4.174)\\ W\hat{}dW &= -dU\hat{}dW. && (4.175) \end{aligned}$$

Subject to these constraints it follows that the process that created the work 1-form is thermodynamically reversible:

$$Q\hat{}dQ = (W + dU)\hat{}dW = W\hat{}dW + dU\hat{}dW = -dU\hat{}dW + dU\hat{}dW = 0. \tag{4.176}$$

The functions $\phi$ and $\chi$ are completely arbitrary. The quantities $\{-U + \phi\chi\}$ are defined as the thermodynamic potentials for specific choices of $\phi$ and $\chi$. For example, the classic choices for the Energy potentials are:

$$\begin{aligned} \{-U + \phi\chi\} &= -U \quad \text{Internal Potential.} \qquad \phi = 0, \chi = 0, \\ \{-U + \phi\chi\} &= TS - U \quad \text{Helmholtz .} \qquad \phi = T, \chi = S, && (4.177)\\ \{-U + \phi\chi\} &= -(PV + U) \quad \text{Enthalpy.} \qquad \phi = -P, \chi = V, && (4.178)\\ \{-U + \phi\chi\} &= -(U - TS + PV) \quad \text{Gibbs Potential.} && (4.179) \end{aligned}$$

In each case the (reversible) work 1-form is given by the formula:

$$W = -dU, \text{ for the Internal Potential.} \tag{4.180}$$
$$W = -d(U-TS) - SdT, \text{ for the Helmholtz Potential.} \tag{4.181}$$
$$W = -d(U+PV) + VdP, \text{ for the Enthalpy Potential.} \tag{4.182}$$
$$W = -d(U-TS+PV) + VdP - SdT, \text{ for the Gibbs Potential} \tag{4.183}$$

The Helmholtz potential is useful for reversible processes for which the temperature is constant, $dT = 0$. The Enthalpy potential is useful for reversible processes for which the pressure is constant, $dP = 0$. The Gibbs potential is useful for reversible processes which involve constant pressure and temperature, $dP = 0$ and $dT = 0$. However, note that the function pair $\phi$ and $\chi$ is completely arbitrary.

Conclusion *The importance of the thermodynamic potentials is their relationship to reversible processes, where the work 1-form is of Pfaff dimension 3, but the Pfaff dimension of the heat 1-form is 2.*

It is important to realize that the thermodynamic potentials so constructed above imply that the contact $2n + 1 = 3$ dimensional structures generated by the work 1-form are "tight" and without limit cycles (see section 3.6.2).

### *4.12.3 Thermodynamic Potentials as Bernoulli evolutionary invariants.*

Under the appropriate conditions of constant pressure, constant temperature, or both, each of the thermodynamic potentials above have the format of Bernoulli functions, $W = i(\rho\mathbf{V}_4)dA = -dB$. Under the constraints of constant temperature or pressure, each of the Potentials is an Bernoulli invariant of the path generated by, $\rho\mathbf{V}_4$, but each potential is not necessarily a global invariant. The proof is easy:

$$L_{(\rho\mathbf{V}_4)}B = i(\rho\mathbf{V}_4)dB = i(\rho\mathbf{V}_4)(L_{(\rho\mathbf{V}_4)}(-W)) = -i(\rho\mathbf{V}_4)(i(\rho\mathbf{V}_4)dA) = 0 \tag{4.184}$$

In other words, depending on the choice of the Bernoulli potential function, $B$, representing the Work 1-form in terms of constrained topological fluctuations, the following evolutionary invariants are determined.

$$L_{(\rho\mathbf{V}_4)}B_{Gibbs} = L_{(\rho\mathbf{V}_4)}(U - TS + PV) = 0, \text{ Gibbs} \quad (4.185)$$

$$L_{(\rho\mathbf{V}_4)}B_{Enthalpy} = L_{(\rho\mathbf{V}_4)}(U + PV) = 0, \text{ Enthalpy} \quad (4.186)$$

$$L_{(\rho\mathbf{V}_4)}B_{Helmholtz} = L_{(\rho\mathbf{V}_4)}(U - TS) = 0, \text{ Helmholtz} \quad (4.187)$$

$$L_{(\rho\mathbf{V}_4)}B_{internal} = L_{(\rho\mathbf{V}_4)}(U) = 0, \text{ Internal (extremal).} \quad (4.188)$$

Hence, the empirical thermodynamic potentials, more than 100 years old in concept, are to be recognized as the Bernoulli-Casimir evolutionary invariants of reversible processes that admit topological fluctuations. These reversible processes can exist on symplectic spaces of topological dimension 2n+2, where the Work 1-form does not vanish, and is Pfaff dimension 3. The need for recognizing the differences between mechanical energy and the thermodynamic energies was discussed by Stuke [230], where, without mention of symplectic evolution, he deduces the need for "acceleration " potentials in certain dissipative systems. These acceleration potentials, which can be shown to be the equivalent of Bernoulli-Casimir functions, were used by Stuke to construct the Enthalpy and Gibbs free energy in certain hydrodynamic examples.

The thermodynamic concepts of pressure and temperature are explicitly absent from that version of classical mechanics which has focused attention on the extremal contact manifolds of dimension $2n + 1$, and which has ignored the concept of topological differential fluctuations on symplectic spaces of dimension 2n+2. It is suggested that the occurrence of a pressure gradient, or a temperature gradient should be taken as the signature of a symplectic process.

On a symplectic domain of dimension $2n + 2$, unique ubiquitous extremal fields of classical Hamiltonian mechanics do not exist. There are no solutions to the extremal equation $i(\mathbf{V})dA = 0$, on the symplectic domain, but there do exist *non unique* vector fields $\mathbf{V}$ that satisfy the Helmholtz constraint equation, $d(i(\mathbf{V})dA) = 0$. In the subset of exact cases, where $i(\mathbf{V})dA = -dB$, these vector fields generate "Hamiltonian-like" dynamical systems, or processes, (on the 2n+1 submanifold transversal to $dB$), similar to the dynamical systems that are associated with the 2n+1 contact manifolds of classical State Space. The Action integral is a relative (stationary) integral invariant with respect to such Hamiltonian dynamical processes. The function $B$ is a Bernoulli-Casimir evolutionary invariant, but these evolutionary invariants (stationary states) are not unique, not independent of gauge conditions, not global constants over the domain, and are strongly dependent upon boundary conditions. The somewhat larger class of vector fields that satisfy the Helmholtz condition $d(i(\mathbf{V})dA) = 0$ are defined as symplectic vector fields, and as dynamical systems they define symplectic processes. However, all such symplectic processes, exact or not, on symplectic domains of dimension 2n+2, still represent *reversible* thermodynamic processes.

Remarkably, and repeated here again for emphasis, on the 2n+2 symplectic domain there exists a *unique* non Hamiltonian vector field which leaves the Action integral a conformal, not stationary, invariant [169]. This unique vector field, defined as the Torsion current, $\mathbf{T}$, does not satisfy the symplectic condition, but instead satisfies the equation, $i(\mathbf{T})dA = \Gamma A$ as suggested in the 1974 article [165]. Moreover, it now can be demonstrated that this unique vector field generates dynamical systems that represent irreversible processes in a thermodynamic sense. This unique vector field (to within a factor) is generated by the formulas

$$A\hat{}(dA)^n = i(\mathbf{T})\Omega_{2n+2\ vol} \tag{4.189}$$

The symplectic space of dimension 2n+2 on which the Torsion current exists is defined as Thermodynamic Space, in order to distinguish it from the classic State Space of dimension 2n+1. The divergence of this Torsion vector field defines a density function on the 2n+2 space. The zero sets of this density function define smooth attractors (inertial manifolds) of dimension 2n+1 on the 2n+2 dimensional domain. The irreversible dynamical system generated by the Torsion vector irreversibly decays to these sets of measure zero which form the "stationary" states of a 2n+1 contact manifold. Once in the stationary state, the evolution can take place by a reversible Hamiltonian process.

## 4.13 Entropy of Continuous Topological Evolution and Equilibrium Submanifolds

A remarkable achievement of a non equilibrium thermodynamics expressed in terms of continuous topological evolution is the ability to formulate a concept of entropy production in an analytic, non phenomenological way - and without the use of statistics. The classic concept of entropy has been extremely hard to define in non statistical terms, for, like potential energy, classical mechanics does not yield a clear visual intuitive picture of "just what is" entropy. Numerous phenomenological constructions have been suggested (such as entropy is a measure of disorder, entropy is the inverse of information, entropy is proportional to area...), but encoding such concepts is difficult. Associated with the concept of entropy is the idea of a system in equilibrium, which at least in approximation is recognized from experience. Cold water poured into a hot bath comes to equilibrium within a perceptibly short time span. On the other hand, inter-change-ability of kinetic energy and potential energy, ultimately yields a visual perception of "energy" related to dynamics, but there seems to be no visual equivalent for "entropy". Moreover, the currently accepted dogma is that entropy always increases on a global scale. These concepts are hard to formulate analytically using physical techniques that have been based upon geometric concepts. It is the purpose of that which follows to demonstrate how a topological, not geometric, point of view enables an analytic coding of the concept of Entropy - without the use of statistics or a phenomenological assumption.

The topological view also gives a mathematical definition of what is meant by an equilibrium physical system. The topological difference between a connected and a disconnected topology is a sufficient topological property which can be used to distinguish an equilibrium system from a non equilibrium physical system. This concept is based on the Frobenius unique integrability theorem, which is valid for both an equilibrium system and an isolated system (of Pfaff topological dimension $\leq 2$), but not for non equilibrium systems (of Pfaff topological dimension $\geq 3$). However, the concept of equilibrium is more subtle. Bamberg and Sternberg [12] suggest that a thermodynamic equilibrium state corresponds to a solution of a Lagrangian submanifold structure to an exterior differential system (in 4D). In 4D, the Lagrangian submanifold of a symplectic manifold generated by a 2-form, $dA$, is a 2 dimensional submanifold upon which the 2-form $dA$ vanishes. Of more interest to this article is how such a submanifold structure may be viewed in terms of the limit set of topological fluctuations in arbitrary dimension. The basic idea is that:

> **Proposition** *Topological Fluctuations lead to a concept of an entropy relative to continuous topological evolution.*

### *4.13.1 Extensions of the Cartan-Hilbert Action 1-form*

This subsection considers in more detail those physical systems that can be described by a Lagrange function $L(\mathbf{q}, \mathbf{v}, t)$ and a 1-form of Action given by:

$$A = L(\mathbf{q}^k, \mathbf{v}^k, t)dt + \mathbf{p}_k \cdot (d\mathbf{q}^k - \mathbf{v}^k dt), \tag{4.190}$$

The classic Action, $L(\mathbf{q}^k, \mathbf{v}^k, t)dt$, is extended to included topological fluctuations in the kinematic variables. It is no longer assumed that the equation of Kinematic perfection is satisfied. That is, fluctuations of the topological constraint of kinematic perfection are permitted:

**Topological Fluctuations :**

$$\textbf{in position}\ \ \Delta\mathbf{q} = (d\mathbf{q}^k - \mathbf{v}^k dt) \neq 0, \tag{4.191}$$
$$\textbf{in velocity}\ \ \Delta\mathbf{v} = (d\mathbf{v}^k - \mathbf{a}^k dt) \neq 0, \tag{4.192}$$
$$\textbf{in momenta}\ \ \Delta\mathbf{p} = (d\mathbf{p}_k - \mathbf{f}_k dt) \neq 0. \tag{4.193}$$

These topological fluctuations are not merely functions of time, but can be fluctuations in space and perhaps other parametric variables. Note that the topological fluctuations are not derivatives, but are differentials - the limit process has not been explicitly stated. One particular fluctuation problem is related to the choice of an "observers" origin. For example, in mechanics it is often assumed that the origin is located at the center of mass. Such an approach can lead to imprecision and fluctuations of parameters, such as mass. The only origin that is free from such defects is a *singular* barycentric system, but that cannot be defined with parameters that are positive definite (such as mass). In the singular barycentric system of projective systems, any point can be used as the "origin" for all other points in an equivalent manner.

When dealing with topological fluctuations, the pre-geometric dimension will not be constrained to only 4 independent variables. At first glance it appears that the domain of definition for the Cartan-Hilbert Action 1-form, above, is a 3n+1 dimensional variety of independent base variables, $\{\mathbf{p}_k,\ \mathbf{q}^k, \mathbf{v}^k, t\}$. The reader is warned that $\mathbf{p}$ is NOT constrained to be a jet; e.g., $\mathbf{p}_k \neq \partial L/\partial \mathbf{v}^k$. Instead, the $\mathbf{p}_k$ are considered to be a (set of) Lagrange multiplier(s) to be determined later. Note that the Action 1-form has the format used in the Cartan-Hilbert invariant integral [44], except that herein it is not assumed that the $\mathbf{p}_k$ are canonical; that is, $\mathbf{p}_k \neq \partial L/\partial \mathbf{v}^k$ necessarily. Also, it is NOT assumed at this stage that the vector field, $\mathbf{v}$, is a kinematic velocity function, such that $(d\mathbf{q}^k - \mathbf{v}^k dt) \Rightarrow 0$. A classical inference is to assert that topological fluctuations in kinematic velocity, $\Delta\mathbf{q}$, are related to pressure, and topological fluctuations in kinematic acceleration, $\Delta\mathbf{v}$, are related to temperature.

As explained in the previous section, the top Pfaffian for the Cartan-Hilbert Action can be evaluated, and is given by the formula,

**Top Pfaffian 2n+2**

$$(dA)^{n+1} = (n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\}\hat{}\,\Omega_{2n+1}, \tag{4.194}$$

and emphasizes the fact the the topological Pfaff dimension of the Cartan-Hilbert 1-form is $2n+2$, and not the "geometrical" dimension $3n+1$. From the fact that the top Pfaffian represents a $2n+2$ volume element,

$$(dA)^{n+1} \Rightarrow \Omega_{2n+2} = dS\hat{}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt, \tag{4.195}$$

such that the bracketed expression in the formula for the top Pfaffian must reduce to an exact differential, $dS$ :

$$(n+1)!\{\Sigma_{k=1}^n(\partial L/\partial v^k - p_k)dv^k\} = \Sigma_{j=1}^n \hbar k_j dv^j \Rightarrow dS. \tag{4.196}$$

Conclusion *As the 2n+2 form represents a volume element, the coefficient of the top Pfaffian has a representation (to within a factor) as a perfect differential of a function, $S$, which is independent from the $\{p_k, q^k, t\}$.* The differential change of the function $S$ is explicitly dependent upon the differentials of velocity $dv^k$ and the non canonical components of momentum $(\partial L/\partial v^k - p_k)$.

**Definition** The change in entropy due to continuous topological evolution is defined as $dS$, and is given by the expression

$$dS = (n+1)!\{\Sigma_{k=1}^n(\partial L/\partial v^k - p_k)dv^k\}. \tag{4.197}$$

**Definition** *The function $S$ whose differential is the 1-form given in eq. ( 4.197) is defined as the entropy of continuous topological evolution.*

The even dimensional 2n+2 form represents an orientable volume element, and once an orientation has been fixed (say +1), then as continuous evolution is constrained to maintain the volume element and its sign, the change in the entropy, $dS$, must be of one sign. So entropy of topological evolution, if it changes globally, can be only of one sign (chosen to be positive in the historic literature). Also, as $dA$ is presumed to be non degenerate, then the differential, $dS$, can not change sign by continuous topological evolution on the 2n+2 dimensional space.

**Conclusion** *Hence the fact that global changes in entropy of topological evolution must be of one sign $\geq 0$ is an artifact of topological orientability, hence $dS$ represents entropy production.*

Next consider subspaces of the Symplectic 2n+2 space. In particular consider a Lagrangian submanifold, which must be dimension n+1. By definition, on the Lagrangian submanifold (of dimension n+1) of the Symplectic space (of dimension 2n+2), the 2-form $dA$ must vanish. The 2-form can be written as:

$$dA = dS\hat{}dt + \{dp_k - \partial L/\partial x^k dt\}\hat{}(\Delta q^k) \Rightarrow 0. \tag{4.198}$$

Observe that the immersion $\psi$ of the configuration space with differentials $\{dq^1\hat{}..dq^n\hat{}dt\}$ into the top Pfaffian space $\{dS\hat{}\Omega_{2n+1}\}$, defines a Lagrangian submanifold when the pullback of the 2-form $dA$ vanishes. The 2-form $dA$ has expression given by the equation above. Consider the case where the immersion into the 3n+1 space is such that the pullback of $(\Delta q^k) \Rightarrow 0$.

$$\psi : (q^1, .., q^n, t) \Rightarrow (S(q, p, t, v), p_1, ..., p_n, q^1, .., q^n, v^1, .., v^n, t) \tag{4.199}$$

Then the 2-form has a pullback realization such that

$$\psi^*(dA) = dS\hat{}dt \Rightarrow 0 \text{ for a Lagrange submanifold.} \tag{4.200}$$

The Pfaff topological dimension of the constrained 1-form of Action is then 2 on configuration space, and induces a connected Cartan Topology. The 2-form vanishes when the entropy is a constant:

**Conclusion** *Equilibrium implies* $dS_{equil}(q, p, t, v) \Rightarrow dS_{equil}(t) \Rightarrow 0$.

It is also remarkable to note that if the momenta are canonically defined, such that

$$\{\partial L/\partial v^k - p_k\} \Rightarrow 0 \supset dS = \{\Sigma_{k=1}^n(\partial L/\partial v^k - p_k)dv^k\} \Rightarrow 0, \tag{4.201}$$

then the entropy production, $dS, vanishes.$The concept of an entropy of continuous topological evolution is explicitly dependent upon the existence of *non canonical* momenta.

It is remarkable that the symplectic systems of irreducible topological dimension $2n+2$ seem to resolve the Boltzmann - Loschmidt-Zermelo paradox of why *canonical* Hamiltonian mechanics is not able to describe the decay to an equilibrium state, and why the usual (extremal) methods of Hamiltonian mechanics do not give any insight into the concept of Pressure, Temperature, Entropy or the Gibbs free energy. It is extraordinary that answers to these

150 year old paradoxes of physics seem to follow without recourse to statistics if one utilizes a topological perspective. The interpretation of the fact that the top Pfaffian (for a physical system that can be encoded by a Cartan-Hilbert 1-form of Action) is of dimension $2n+2$ and not $3n+1$ is, at present, not complete. The implication is that there must exist $(3n+1)-(2n+2) = n-1$ topological invariants in these systems.

Consider a process starting with some initial conditions in the turbulent domain of Pfaff dimension 4 for $A$, $W$, and $Q$. If the process proceeds by evolution such that the process path enters a region of the geometric domain where either $T \Rightarrow 0$, or the Topological Entropy production rate vanishes by orthogonality,

$$\textbf{Orthogonality}: \{\Sigma_{j=1}^{n} \hbar k_j dv^j\} = dS_{\mathbf{v}} \Rightarrow 0, \tag{4.202}$$

or, if a domain is reached such that the momenta become canonical

$$\textbf{Canonical Momenta}: p_k = \partial L/\partial v^k, \tag{4.203}$$

it follows that the Cartan-Hilbert Action, $A$, decreases its topological dimension from $2n+2$ to $2n+1$. This $2n+1$ Contact manifold is the state space of classical mechanics. When that Action 1-form generates a Contact manifold, there is always a unique extremal vector field which generates a system of first order ODE's known as Hamilton's equations describing the extremal process. If at subsequent steps of the evolutionary path all of the differentials $dp_k$ become zero, then the dimensionality of the 2n+2 manifold becomes the configuration space manifold of dimension n+1, a LaGrangian submanifold. If the Pfaff dimension of $A$ is equal to 1 when $A$ is restricted to the submanifold, the equilibrium state has been defined in which the entropy function, $S$, is a constant; e.g., $dS_{\mathbf{v}} = 0$.

The important facts are that there are two classes of processes that can represent the topological change from a Pfaff topological dimension 2n+2 to a Pfaff topological dimension of 2n+1. The 2n+2 system supports thermodynamically irreversible dissipative processes. The 2n+1 system supports stationary reversible Hamiltonian processes. The two classes of processes are distinguished by the property that the velocity field is orthogonal to the non canonical momenta, or the process causes the non canonical momenta to vanish.

If the domain of definition is constrained such that the momenta are defined canonically, $\partial L/\partial v^k - p_k = 0$, then the 2-form $dA$ does not define a symplectic manifold of Pfaff topological dimension 2n+2, but the 2-form does define Contact structure on 2n+1 with the formula for the Top Pfaffian given by the expression.

**Top Pfaffian 2n+1**

$$A\hat{}(dA)^n = n!\{p_k v^k - L(t, q^k, v^k)\} dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt. \quad (4.204)$$

The coefficient in brackets is recognized as the Legendre transform of the Lagrangian producing the format of the classic Hamiltonian energy. It is this 2n+1 dimensional contact manifold that served as the arena for most of classical mechanics prior to 1955, especially for those theories which were built from the calculus of variations. The 2n+1 dimensional contact manifold, or state space, admits a unique "extremal" evolutionary field, $i(\mathbf{V})dA = 0$, that satisfies "Hamilton's equations". The coefficient of the state space volume is to be recognized as the Legendre transform of the physicist's Hamiltonian energy function.

$$L(t, q^k, v^k) = p_k v^k - H(t, q^k, v^k, p_k) \quad (4.205)$$

When the constraints of canonical momenta are valid, it follows that

$$\partial H(t, q^k, v^k, p_k)/\partial v^k = 0. \quad (4.206)$$

This result is interpreted by the statement that the "Hamiltonian" is to be expressed in terms of the variables $\{t, q^k, p_k\}$ only. The Top Pfaffian becomes

$$A\hat{}(dA)^n = n!\{H(t, q^k, p_k)\} dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt. \quad (4.207)$$

The 2n+1 space maintains its contact structure as long as the "total Hamiltonian energy" is never zero, and the momenta are canonically defined.

If further topological evolution causes the Pfaff topological dimension to change from 2n+2 to 2n, then it follows that the Hamiltonian energy must vanish. That is (using the canonical constraint), reduction of the Pfaff dimension from 2n+1 to 2n (state space to phase space) requires that the LaGrange function be homogeneous of degree 1 in the velocities, $v^k$:

**Top Pfaffian 2n**

$$(dA)^n = \{v^k \partial L(t, q^k, v^k)/\partial v^k - L(t, q^k, v^k)\} \Rightarrow 0. \quad (4.208)$$

The result is remarkable in that the definition of a Finsler space is precisely that constraint situation where the coefficient $\{p_k v^k - L(t, q, v\}$ of the 2n+1 manifold vanishes, and the momenta are canonical. These constraint of canonical momentum and that the constraint that the Lagrangian be homogeneous of degree 1 in the velocities are precisely Chern's constraints. Chern uses these constraints to define a Finsler space [45] which admits non Riemannian geometries (when the Lagrange function contains more than quadratic powers of $\mathbf{v}$) and spaces with torsion [28]. Note that the processes of topological reduction described above are not equivalent to forming an arbitrary section(s) in the form of holonomic constraints.

# Chapter 5

# NOTES AND REFERENCES

## 5.1 Notes

Many *topics* can be found on the web via the ending portion of the URL ../*topic*.pdf (as noted in the bibliography below), and attached to the URL prefix http://www22.pair.com/csdc/pdf to form:

**http://www22.pair.com/csdc/pdf/topic.pdf**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## 5.2 About the Cover Picture


This picture demonstrates the existence of Falaco Solitons which are rotational topological defect structures created by irreversible decay to long lived states far from equilibrium. The Solitons are not locally stabilized, but are globally stabilized by the tension in a string connecting the vertices of the minimal surface dimples. The universal topological defect appears in many different disguises:

If the dimpled surfaces are defined as 2D Branes, and the connecting thread as a 1D string, then the Falaco Solitons are realizations of string theory, but not at a cosmological scale.

If the dimpled surfaces are defined as quark pairs, and the connecting thread as the confinement mechanism, then the Falaco Solitons are realizations of Quark theory, but not at a sub-microscopic scale.

The Falaco Solitons are thermodynamically long lived states far from equilibrium.

## 5.3 About the Author

Professor R. M. Kiehn, B.Sc. 1950, Ph.D. 1953, Physics, Course VIII, MIT, started his career working (during the summers) at MIT, and then at the Argonne National Laboratory on the Navy's nuclear powered submarine project. Argonne was near his parents home in the then small suburban community known as Elmhurst, Illinois. At Argonne, Dr. Kiehn was given the opportunity to do nuclear experiments using Fermi's original reactor, CP1. The experience stimulated an interest in the development of nuclear energy. After receiving the Ph. D. degree as the Gulf Oil Fellow at MIT, Dr. Kiehn went to work at Los Alamos, with the goal of designing and building a plutonium powered fast breeder reactor, a reactor that would produce more fissionable fuel then it consumed. He was instrumental in the design and operation of LAMPRE, the Los Alamos Molten Plutonium Reactor Experiment. He also became involved with diagnostic experiments on nuclear explosions, both in Nevada on shot towers above ground, and in the Pacific from a flying laboratory built into a KC-135 jet tanker. He is one of the diminishing number of people still alive who have witnessed atmospheric nuclear explosions.

Dr. Kiehn has written patents that range from AC ionization chambers, plutonium breeder reactor power plants, to dual polarized ring lasers and down-hole oil exploration instruments. He is active, at present, in creating new devices and processes, from the nanometer world to the macroscopic world, which utilize the features of Non Equilibrium Systems and Irreversible Processes, from the perspective of Continuous Topological Evolution.

Dr. Kiehn left Los Alamos in 1963 to become a professor of physics at the University of Houston. He lived about 100 miles from Houston on his Pecan Orchard - Charolais Cattle ranch on the banks of the San Marcos river near San Antonio. As a pilot, he would commute to Houston, and his classroom responsibilities, in his Cessna 172 aircraft. He was known as the "flying professor".

He is now retired, as an "emeritus" professor of physics, and lives in a small villa at the base of Mount Ventoux in the Provence region of southeastern France. He maintains at active scientific website at

http://www.cartan.pair.com

# Non Equilibrium Systems and Irreversible Processes

R. M. Kiehn

**Adventures in Applied Topology Series:**

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