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1 Longitudinal modes of a laser cavity

1.1 Resonant modes

For the moment, imagine a laser cavity as a set of plane mirrors separated by a distance d. We will return to the specific properties of the cavity later.

Resonant modes of the laser cavity are those that are capable of producing standing waves. These modes occur at the wavelengths λ_n that satisfy

$$2d = n\lambda_n$$

The possible resonant frequencies of the cavity are

$$\nu_n = \frac{c}{\lambda_n} = \frac{nc}{2d}$$

The resonant frequencies of the cavity are equally spaced,

$$\nu_{n+1} - \nu_n = \frac{c}{2d}$$

1.2 Laser gain parameter

In considering the output of a laser, we defined the laser gain parameter,

$$\alpha_\nu = \frac{h}{c}\frac{\nu}{\Delta\nu}B_{21}\left(\Delta N_2 - \Delta N_1\right)$$

where $\Delta N_{1,2}$ is the concentration of particles in the state with energy $E_{1,2}$ in the frequency range $\Delta\nu$, and $\Delta\nu$ is the range of frequencies in the beam output by the laser.

In order to achieve lasing, the energy loss due to absorption in each round trip in the laser cavity must be less than the gain obtained in each trip. We quantified this condition using a relative efficiency of the laser, δ, where $\delta \leq 1$, which represents the energy loss due to absorption by the laser cavity. We saw that in order to achieve lasing, it must be that,

$$e^{2\alpha_\nu d} > \frac{1}{\delta_\nu}$$

Consider the plot of α_ν as a function of frequency. α_ν has a maximum at the center of the laser bandwidth, $\nu = \frac{E_2 - E_1}{h}$. The lasing threshold describes a horizontal line at the value of $\alpha_\nu = -\frac{\ln \delta_\nu}{2d}$. Lasing is only possible for values of α_ν above this line. The possible lasing frequencies are equidistant in frequency space at intervals of $\frac{c}{2d}$.

1.3 Hole burning

Recall that α_ν was involved in the solution to the differential equation that gave the irradiance of a beam as a function of distance traveled in the lasing medium,

$$I(z) = I(0)e^{\alpha_\nu z}$$

If α_ν is above the lasing threshold, this equation appears to predict that $I(z)$ will increase without bound. In practice, the irradiance is limited by the number of atoms in the lasing medium available to undergo spontaneous emission. As atoms are emitted from the excited state by stimulated emission, there are fewer atoms in the excited state near the lasing frequencies. These depopulations appear as dips in the absorption spectrum of the lasing medium near the lasing frequencies. This depopulation effect is known as *hole burning*.

Note. The "hole" being burned is in frequency space only, and does not refer to any tangible hole.

The finite time for repopulation of the pumped state leads to some natural frequency width in output of the laser,

$$\Delta\nu_{\text{nat}} = \frac{E_2 - E_1}{\Delta t}$$

where Δt is the lifetime of the excited state.

1.4 Doppler broadening

Since the gain is inversely proportional to $\Delta\nu$, it is advantageous to make the bandwidth of the laser as small as possible. However, $\Delta\nu$ is fundamentally limited by *Doppler broadening.* This effect occurs because the atoms producing the radiation used by the laser apparatus have a velocity due to their thermal energy. Motion of any source of radiation, in this case atoms of the lasing medium, results in a Doppler shift in the outgoing radiation. Because the thermal motion of the atoms is random, the magnitude of the Doppler shift in the radiation emitted by each atom is different, leading to a spread in output frequencies.

The frequency shift induced by Doppler broadening is characterized by $\frac{\Delta\nu}{\nu}$. Let u_x denote the thermal velocity of an atom in the lasing medium. Using the Boltzmann factor, which leads to the equipartition theorem,

$$\begin{aligned} f(\mu_x) &\propto e^{-\frac{\frac{1}{2}mu_x^2}{kT}} \\ \Rightarrow \frac{1}{2}m\left\langle u_x^2\right\rangle &= \frac{1}{2}kT \\ \Rightarrow \frac{\left\langle(\Delta\nu)^2\right\rangle}{\nu^2} &= \frac{kT}{mc^2} \end{aligned}$$

At room temperature,

$$kT \sim \frac{1}{40}\text{ eV} \sim O(10^{-3}\text{eV})$$

The rest energy is

$$mc^2 \sim O(\text{GeV}) = O(10^9\text{eV})$$

The relative magnitude of Doppler broadening is,

$$\begin{aligned} \frac{\left\langle(\Delta\nu)^2\right\rangle}{\nu^2} &\sim O(10^{-12}) \\ \Rightarrow \frac{\left\langle\Delta\nu\right\rangle}{\nu} &\sim O(10^{-6}) \end{aligned}$$

1.5 Line-shape of α_ν

Due to the effects of hole burning and Doppler broadening, the line-shape of α_ν is not Gaussian. Instead, it is proportional to,

$$\alpha_\nu \propto \frac{1}{\Delta\nu^2 + \frac{\Gamma^2}{(4\pi)^2}}$$

where Γ is the inverse mean life for the transition E_2 to E_1.

2 Transverse modes of a laser

2.1 Motivation

Intuitively, we expect the irradiance pattern of a laser beam to be circular; if the beam is shined on a material, we expect to observe a circular hole burned on the material. However, we will see that it is possible to obtain other transverse modes, for which the resulting irradiance patterns are not circular. We will first study the origin of these effects. We will then see how they can be removed, to ensure a highly focused output beam.

2.2 Analogy to Fraunhofer diffraction

Imagine a simple setup in which there are two concave mirrors with focal length f, which implies that their radii of curvature are $R = 2f$. Place these mirrors confocally, with the concave sides of the mirrors facing each other. Imagine a beam beginning at the point $z = 0$, reflecting off each mirror, and returning to the point $z = 0$. The total length of this path is $4f$.

Using some hand waving, we can say that this setup is very similar to the previous setup that we considered when studying Fraunhofer diffraction. The path length of the apparatus in that case was also $4f$, where f was the focal length of the thin field lenses placed between the aperture, transform, and image planes.

We saw in the case of Fraunhofer diffraction that the optical disturbance at the image plane is the fourier transform of that at the transform plane. In this case, the concave mirrors function as the thin lenses, and so the transform and image planes are both located at the center point of the two confocal concave lenses.

After many reflections of the beam, we expect by symmetry that the optical disturbance at $z = 0$ should be the same before and after reflection off one of the mirrors. Combined with the previous result, this implies that the optical disturbance must be a function that is equal to its Fourier transform.

2.3 Possible solutions

One possibility is the Gaussian, which is equal to its Fourier transform as we mentioned previously,

$$U(x,y) = e^{-\frac{(x^2+y^2)}{w^2}}$$

where w is a characteristic width.

There are also other higher order solutions,

$$U_{pq}(x,y) = H_p\left(\frac{\sqrt{2}x}{w}\right) H_q\left(\frac{\sqrt{2}y}{w}\right) e^{-\frac{(x^2+y^2)}{w^2}}$$

where $H_{p,q}$ are Hermite polynomials,

$$H_0(u) = 1 \qquad H_1(u) = 2u \qquad H_n(u) = (-1)^n e^{u^2} \frac{d^n}{du^n}\left(e^{-u^2}\right)$$

Note.

- With this notation, the Gaussian case can be denoted U_{00}.
- These are the same Hermite polynomials that appear in the solution to the 1D Shrödinger equation for the quantum mechanical simple harmonic oscillator.

We are considering the case in which the electromagnetic fields are perpendicular. In this case, the solutions are called *transverse electromagnetic* (TEM) solutions.

Each choice of p and q leads to a different irradiance pattern. Some examples are:

TEM$_{00}$ consists of a pure circle

TEM$_{10}$ consists of two symmetric spots

TEM$_{11}$ consists of four spots, the total forming a circular shape with the coordinate axes removed from the circle.

Visualizations of some of these patterns appear in *Padrotti*, figures 22-17 and 22-18.

All TEM modes beyond the 00 mode produce an undesirable spread in the irradiance of the output laser beam. We would like to eliminate all of these higher order modes to create *single mode laser* with maximum point-like irradiance.

3 Description of the beam field

3.1 Motivation

Our discussion thus far is in terms of U, a scalar function. To return vector fields to the problem, assume all fields are polarized in the $\hat{\mathbf{x}}$ direction.

Our discussion of Fraunhofer diffraction also assumed plane waves. The waves emitted by a laser have a very limited transverse extent and are certainly not plane waves. To better understand the case of waves comprising the laser beam, we must return to the underlying physics encapsulated in Maxwell's equations.

3.2 Calculation

We consider the case of an insulator, which will simplify the resulting equations. In this case, the conductivity σ is equal to 0, and there is no attenuation of the beam within the material. Maxwell's equations become,

$$\left(\nabla^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = 0$$

Going from the vector to the scalar case, assume $\hat{\mathbf{x}}$ polarization,

$$\mathbf{E}(\text{phys}) = \hat{\mathbf{x}}E(\text{phys})$$

where the physical field is, as usual, the real part of a complex field. Assume a monochromatic wave, so that the fast timescale variation of the wave can be expressed as $e^{-\imath\omega t}$. The complex field contains the slower variations in the field,

$$E(\text{phys}) = \Re\left(E(\mathbf{r})e^{\imath(kz-\omega t)}\right)$$

$E(\mathbf{r})$ is dependent on z, although the variation in the complex field is slow relative to the exponential dependence on z. $E(\mathbf{r})$ is dependent on x and y, and in particular, has important dependence on $\rho = \sqrt{x^2 + y^2}$.

Now, substitute this assumed form of the $\mathbf{E}$ field into Maxwell's equation above. Computing the derivatives,

$$\begin{aligned}
\nabla^2 \left(E(\mathbf{r})e^{\imath(kz-\omega t)}\right) &= \boldsymbol{\nabla}\left(\boldsymbol{\nabla}E(\mathbf{r})e^{\imath(kz-\omega t)} + E(\mathbf{r})\boldsymbol{\nabla}e^{\imath(kz-\omega t)}\right) \\
&= \nabla^2 E(\mathbf{r})e^{\imath(kz-\omega t)} + \boldsymbol{\nabla}E(\mathbf{r})\cdot\boldsymbol{\nabla}e^{\imath(kz-\omega t)} \\
&+ \boldsymbol{\nabla}\mathbf{E}(\mathbf{r})\cdot\boldsymbol{\nabla}e^{\imath(kz-\omega t)} + E(\mathbf{r})\nabla^2 e^{\imath(kz-\omega t)} \\
\frac{\partial^2}{\partial t^2}\left(E(\mathbf{r})e^{\imath(kz-\omega t)}\right) &= E(\mathbf{r})\frac{\partial^2}{\partial t^2}e^{\imath(kz-\omega t)} \\
\Rightarrow \left(\nabla^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}\right)\left(E(\mathbf{r})e^{\imath(kz-\omega t)}\right) &= \nabla^2 E(\mathbf{r})e^{\imath(kz-\omega t)} + 2\boldsymbol{\nabla}E(\mathbf{r})\boldsymbol{\nabla}e^{\imath(kz-\omega t)} \\
&+ E(\mathbf{r})\left(\nabla^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}\right)e^{\imath(kz-\omega t)}
\end{aligned}$$

The derivatives of the exponential can be simplified, assuming a particular value for k,

$$\begin{aligned}
\left(\nabla^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}\right)e^{\imath(kz-\omega t)} &= \left(-k^2 + \epsilon\mu\omega^2\right)e^{\imath(kz-\omega t)} \\
\Rightarrow \left(\nabla^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}\right)e^{\imath(kz-\omega t)} &= 0 \text{ if } k = \omega\sqrt{\epsilon\mu} = \frac{\omega}{c}\sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} = \frac{\omega}{v_{\text{ph}}}
\end{aligned}$$

The second term in the expression above can also be simplified,

$$\begin{aligned}
\boldsymbol{\nabla}e^{\imath(kz-\omega t)} &= \imath k e^{\imath(kz-\omega t)}\hat{\mathbf{z}} \\
\boldsymbol{\nabla}E(\mathbf{r}) &= \frac{\partial E(\mathbf{r})}{\partial x}\hat{\mathbf{x}} + \frac{\partial E(\mathbf{r})}{\partial y}\hat{\mathbf{y}}\frac{\partial E(\mathbf{r})}{\partial z}\hat{\mathbf{z}} \\
\Rightarrow \boldsymbol{\nabla}E(\mathbf{r})\cdot\boldsymbol{\nabla}e^{\imath(kz-\omega t)} &= \imath k\frac{\partial E(\mathbf{r})}{\partial z}e^{\imath(kz-\omega t)}
\end{aligned}$$

Canceling the remaining exponential factors in the Maxwell equation,

$$\nabla^2 E + 2\imath k\frac{\partial E}{\partial z} = 0$$

As mentioned above, The variation of $\nabla^2 E$ is slow with respect to $e^{\imath kz}$. This we can neglect $\frac{\partial^2 E}{\partial z^2}$ with respect to $k\frac{\partial E}{\partial z}$. Hence we can neglect the $\frac{\partial^2}{\partial z^2}$ component of the ∇^2 operator, although the variation in the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components is still significant,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2\imath k\frac{\partial}{\partial z}\right)E(\mathbf{r}) = 0 \tag{1}$$