Numerical Simulation of Acoustic Wave Scattering with Double Layer Potential Approach

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August 28, 2009

Abstract

In this article, I use the already fully developed double-layer approach to solve the scattering problem of acoustic wave in two dimensions. Numerical Experiments are carried with different wave numbers and in distinct spatial regions. The properties of the scattering waves with the long and small wavelength limits are observed both in the near and far scattering zones. By analog with electromagnetic waves, I give some physical interpretations for these properties.

Introduction

First of all, the total waves outside an obstacle satisfy the wave equation:

$$\frac{\partial^2 U}{\partial t^2} = c^2 \nabla^2 U.$$

where U can be interpreted as the amplitude of the wave. For $U = U(x,t)$ varying in time, one can make a Fourier analysis of $U(x,t)$ and handle each Fourier component separately. In this sense, only the ω component $u(x)e^{-i\omega t}$ is considered in the following analysis without losing generality. Helmholtz equation in frequency domain is obtained by substituting $U(x,t) = u(x)e^{-i\omega t}$ into the wave equation:

$$\nabla^2 u + k^2 u = 0. \tag{1}$$

where the wave number $k = \omega/c$.

For the electromagnetic scattering system, when a plane wave $u^i(x,t) = e^{i(kx-\omega t)}$ is incident on the scatterer, it will induce electrical and magnetic multipoles and cause the multipoles to oscillate with the frequency of ω. As is known, time-varying charges and currents will radiate. The scattering wave is then produced by the radiating systems of electrical and magnetic multipoles. This is the process of electromagnetic scattering. The acoustic scattering works the same way just replacing the radiation with mechanical oscillation.

The total wave $u = u^i + u^s$. It is easy to verify that the incident plane wave u^i satisfy Equ. 1, so the scattering wave should also satisfy the Helmholtz equation. Also, in this article, I deal with the Dirichlet boundary conditions, that is $u|_{\partial D} = 0$. Therefore the problem of calculating scattering waves leads to the exterior Dirichlet Helmholtz equation:

$$\begin{cases} \nabla^2 u^s + k^2 u^s = 0, \; x \in R^2 \backslash D \\ u^s|_{\partial D} = -u^i. \end{cases}$$

The difficulty of this problem stems from the unbound computation domain. There are many approaches that can deal with unbounded regions, such as finite element method or finite different method with perfectly matched layers. In this article, I adopt the double-layer approach to solve the scattering problems.

The double-layer potential is the following equation:

$$u^s(x) = -u^i(x) = \frac{1}{2}\varphi(x) + \int_{\partial D} \varphi(y) \frac{\partial G(x,y)}{\partial n(y)} ds(y), \; x \in \partial D.$$

The integral is on the boundary of the scatter and G(x,y) is the Green function of Helmholtz equation:

$$G(x, x') = \begin{cases} \dfrac{e^{ik|x-x'|}}{|x-x'|} & in\ 3D \\ \dfrac{i}{4} H_0^{(1)}(k|x-x'|) & in\ 2D \end{cases}$$

The incident wave u^i is known and hence all that I have to do is to solve the second kind of Fredholm integral equation numerically to get the unknown function $\varphi(x)$. After that is done, simply by doing the integration:

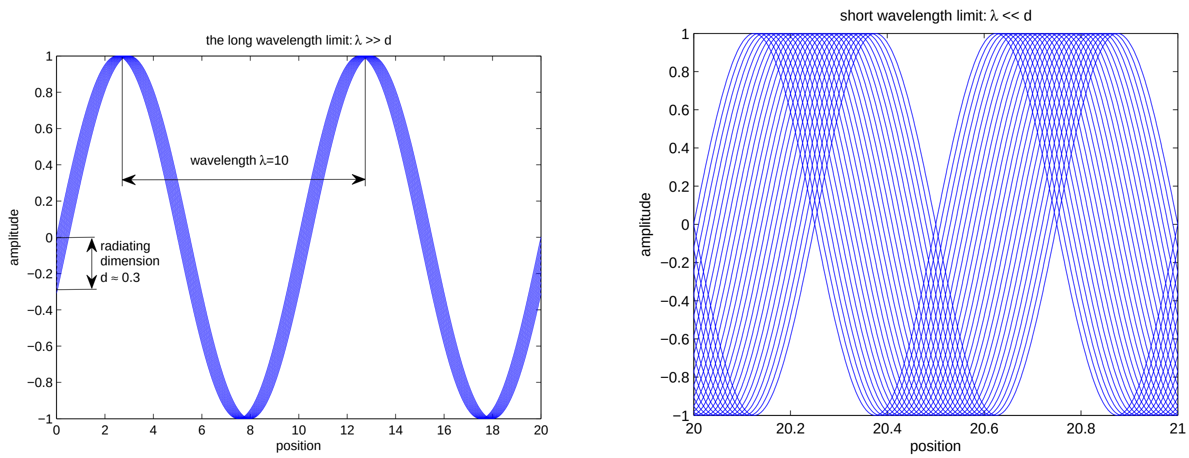
$$u^s(x) = \int_{\partial D} \varphi(y) \frac{\partial G(x,y)}{\partial n(y)} ds(y), \; x \in R^2 \backslash D.$$

I can get the scattering wave in the whole exterior region.

Details for the double-layer approach and its numerical implementation, please refer to 1,2.

Numerical Results and Discussions

Before getting started to the numerical simulation, I think it is necessary for me to give some general properties for the scattering wave in physics. I'll take the electromagnetic wave for example. Clearly, the radiating field is closely related with the relative length scales of the radiating waves. Assume there are many small radiating particles within the scatterer, then the scattering wave is just the superposition of waves generated by each small particle. Obviously, each wave are different from another in phase and polarization. If the wavelength is large compared to the scatterer, the difference in phase and polarization are trivial compared to the scale of wave and hence can be neglected, as illustrated in Figure 1a, in this case, the radiating particles can be regarded as a whole, i.e., the electrical and magnetic dipoles. However, if the wavelength is small compared to the size of target, the distribution of radiating particles play an important role in determining the total wave and therefore the waves will interfere with each other. In this way, the radiating systems can not be considered as a dipole and must be treated as separate ones that is multipoles. An good physical example that can illustrate the influence of the relative length scale of the wave is that suppose you throw two stones into a river, if the position of the two stones are near enough, then the water wave caused by them can be regarded as one. But if the they are separate, the interference effect of these two radiating source must be considered.



(a) (b)

Figure 1: illustrations for the influence of the relative length scale of wave

Another thing that should be taken care is the region of interest. It is no doubt that the scattering field will have distinct properties in different spatial domains. In the far or radiation zone, the scatterer is like a point radiating souse. If I take two spheres with radius r_1 and r_2 encapsulating the point source, one can expect that the total energy flux through these two spheres are equal, that is, $u(r_1)^2 \cdot 4\pi r_1^2 = u(r_2)^2 \cdot 4\pi r_2^2$, and hence I get $u(r) \sim 1/r$. I should notice that this relation between the amplitude u and distance r is valid as long as the radiating systems can be regarded as a point source, no matter the relative scales of the scattering wave. For the short wavelength limit, because the scattering waves are severely interfered, the relationship is right only in the average sense, i.e., $< u(r) >$ over all angles. In the near or static zone, it can be expected that the scattering waves are mainly dependent on the shape and distribution of radiating systems rather than the incident waves.

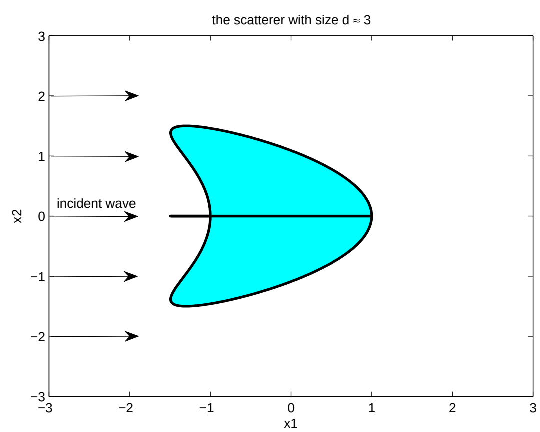


Figure 2: the shape of scatterer

In my simulation, I use the kite-like region as the scatter, as is shown in Figure 2. the parametric equations of this region is:

$$\begin{cases} x(t) = \cos t + 0.65 \cos 2t - 0.65 \\ y(t) = 1.5 \sin t \end{cases}$$

The size of the target is approximated as $d \approx 3$. The incident wave is the plane wave $u^i(x) = e^{ik\vec{n}\cdot\vec{x}}$, where $\vec{n}$ is the incident direction and $(1, 0)$ is used in my simulation as is shown in Figure 2.

First, I consider the scattering with different incident wavelengths. Three numerical experiments are carried with long wavelength $\lambda = 60 >> d$, moderate wavelength $\lambda = 3 \sim d$ as well as short wavelength $\lambda = 0.2 << d$. Different properties are observed in the three cases. Figure 3 shows the long wavelength case. The waves do not interfere with each other and spread out uniformly. Figure 4 is the short wave length case, from which you can see the severe

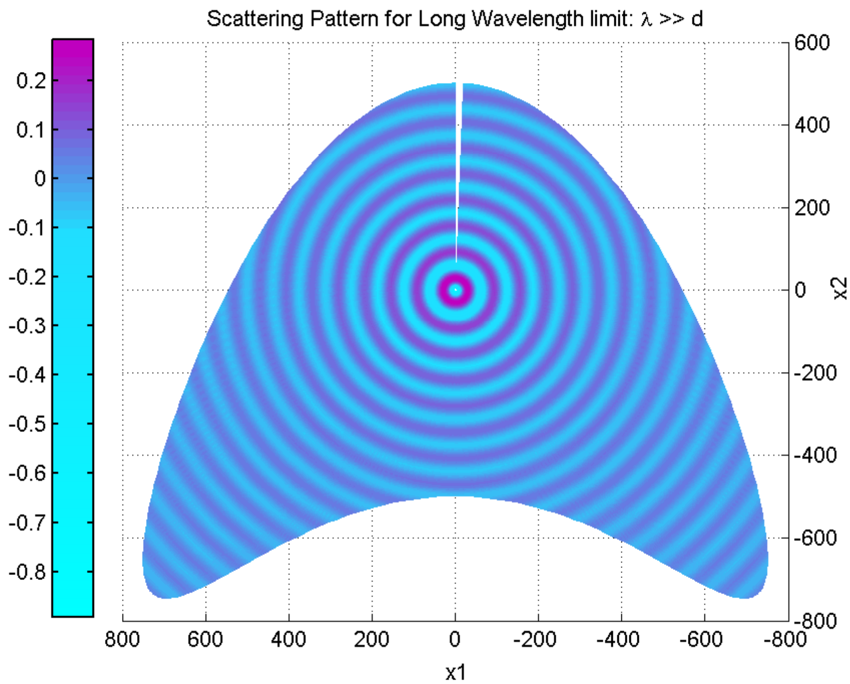


Figure 3: long wavelength

interference. Moreover, at the $(0, 2)$ point, you can see there is a brighter region just like the incident wave focusing there in the geometry optic sense. The simulation result in the long ans short wavelength limits are in agreement with my physical interpretation above. But for the moderate wavelength, Figure 5, the intensity of the scattering waves along the x1-axis are greater than that elsewhere.

In the next step, I would like to demonstrate the properties in a more rigorous way. Again, I will take electromagnetic wave for example. With the Lorenz gauge $\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$, the Maxwell equation can be reduced to:

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \tag{2}$$

Solve Equ. 2 with Green's function one can get:

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}d^3x'. \tag{3}$$

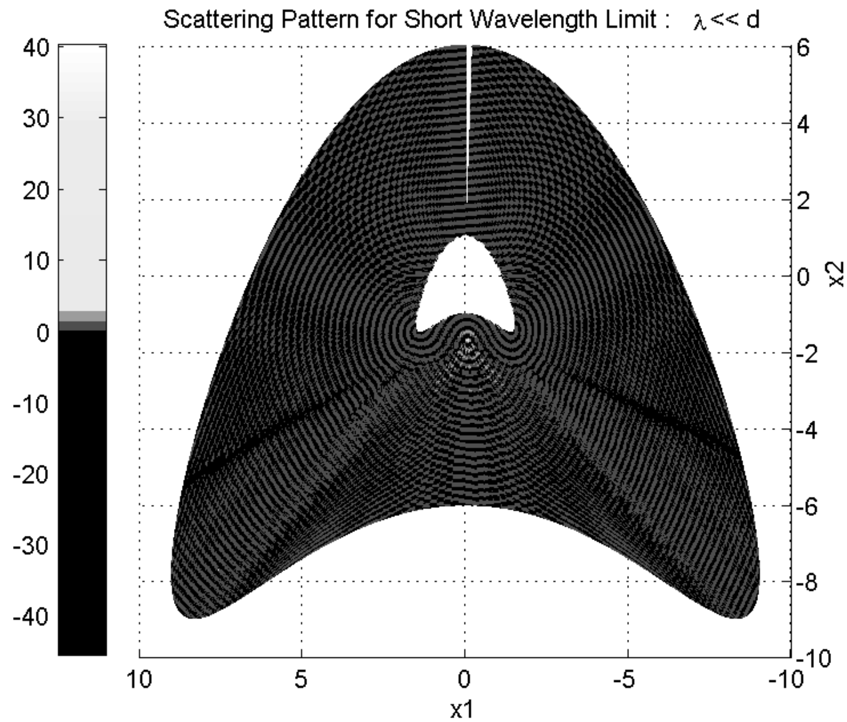


Figure 4: short wavelength

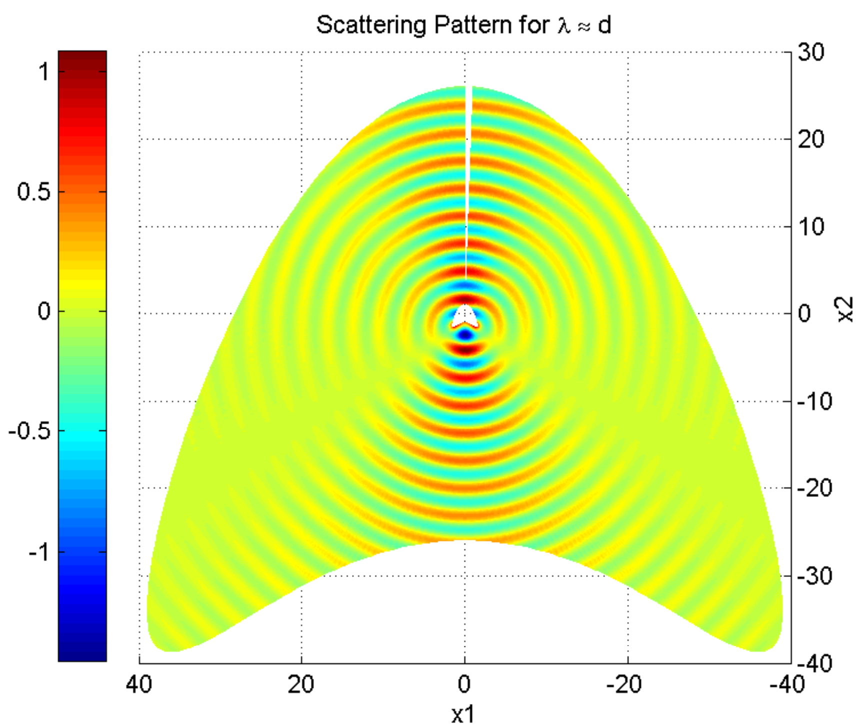


Figure 5: moderate wavelength

In the long wavelength, $kd << 1$, the term $k|\mathbf{x}-\mathbf{x}'|$ in Equ. 3 can be approximated as:

$$\begin{aligned} k|\mathbf{x}-\mathbf{x}'| &= k|\mathbf{x}|(1+(\frac{|\mathbf{x}'|}{|\mathbf{x}|})^2 - 2\frac{\mathbf{x}}{|\mathbf{x}|}\cdot\mathbf{x}')^{\frac{1}{2}} \\ &= |\mathbf{x}|(k - k\mathbf{x}'\cdot\hat{\mathbf{x}} + \frac{1}{2}k|\mathbf{x}'\cdot\hat{\mathbf{x}}|^2 - \cdots). \\ &\approx k(|\mathbf{x}| - \mathbf{x}'\cdot\hat{\mathbf{x}}). \end{aligned}$$

Substitute this approximation into Equ. 3 we get:

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \mathbf{J}(\mathbf{x}')e^{-ik\mathbf{x}'\cdot\hat{\mathbf{x}}}d^3x'.$$

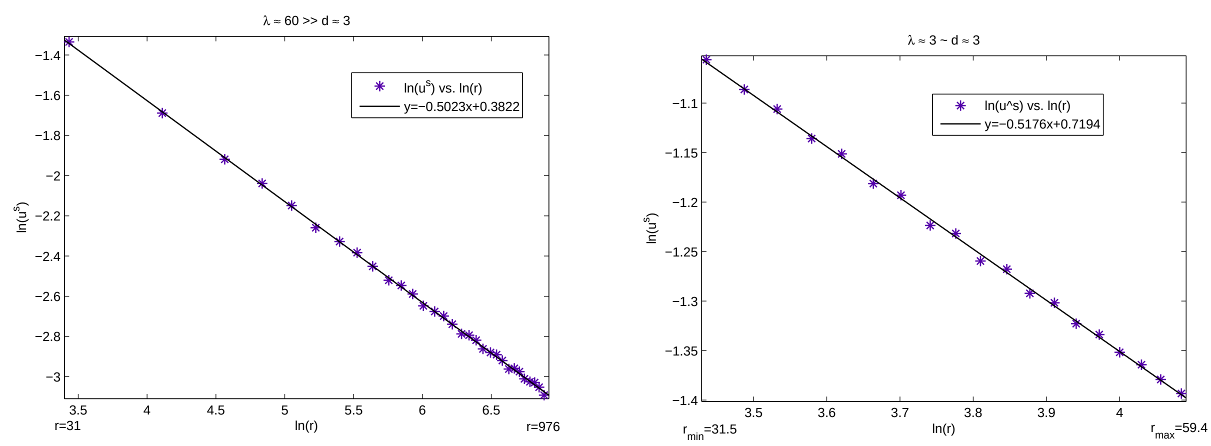
The integral part shows that the scatterer's shape and the current induced by the incident wave can determine the scattering waves while the coefficients in front tells tell that the scattering wave will decay as r^{-1} and oscillate uniformly in space with a phase of kr. This is just what we have observed in Figure 3. Note that the phase term in the integral $e^{-ik\mathbf{x}'\cdot\hat{\mathbf{x}}}$ will not affect the total phase of the scattering wave in the sense that $kd << 1$.

However in the short wavelength limit, the approximation no longer holds, because the higher order terms $\frac{1}{2}k|\mathbf{x}'\cdot\hat{\mathbf{x}}|^2$ cannot be omitted since $kd >> 1$. So the potential is

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \mathbf{J}(\mathbf{x}')e^{-ik\mathbf{x}'\cdot\hat{\mathbf{x}}+\frac{1}{2}ik|\mathbf{x}'\cdot\hat{\mathbf{x}}|^2\cdots\cdots}d^3x'.$$

With the $kd >> 1$, the total phase of the scattering wave is expected to be greatly influenced by the phase term in the integral and therefore make the interference, but the amplitude will still fall off as r^{-1} no matter the wavelength as I mentioned at the beginning this section.

Next I would like to examine the r^{-1} dependence of the amplitude. Since the wave is in two dimension, so relation between amplitude and distance is $r^{-1/2}$ $(u^2(r)2\pi r = u^2(r')2\pi r')$. For long and moderate wavelengths, I take the amplitude of the same phase in the direction of $(1,0)(\pm\pi$, i.e. the peak). Figure 6a,6b show that the slope of the fitting line for $ln(u^s) \sim ln(r)$ is very close to the theoretical value 0.5. But in the short wavelength limit, such dependence is not found because the interference claims the amplitude to be estimated over all directions rather than sample the $(1,0)$ direction.



(a) long wavelength

(b) moderate wavelength

Figure 6: illustrations for the influence of the relative length scale of wave

References

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