Faraday Effect

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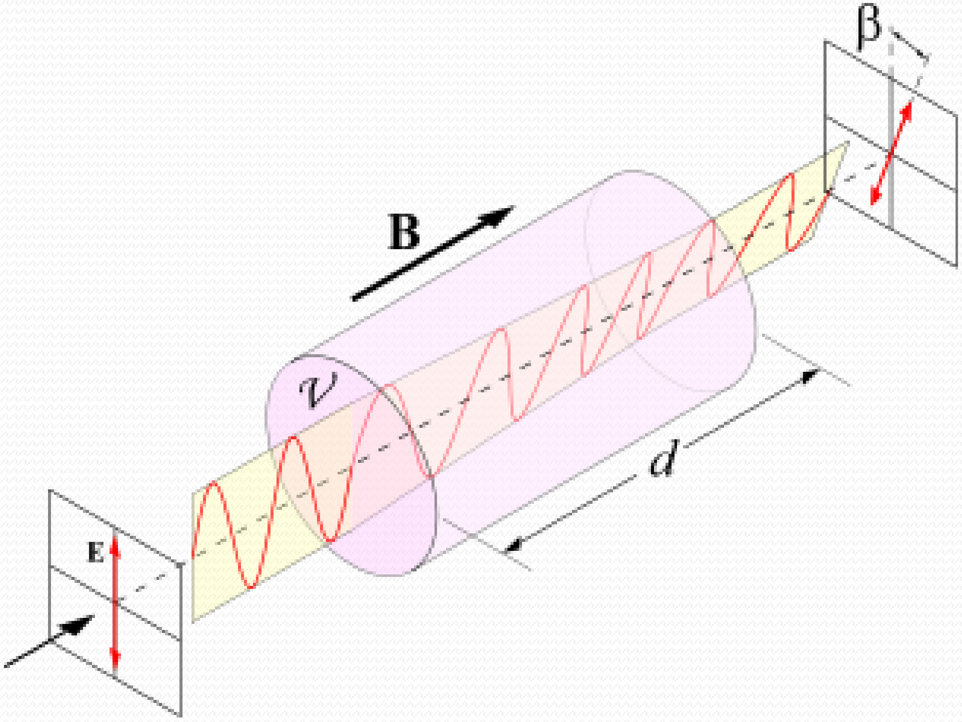
Outline:

- Illumination of Faraday Effect
- Experimental devices
- Analysis
- Unexpected phenomenon

- Measurement of Dispersion
- Analysis

- Appendix
- Faraday Effect in Classical Electrodynamics fashion
- Theory of Dispersion

Basic Phenomenon

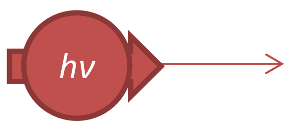


From Wikipedia

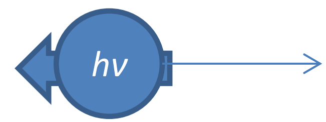
An Intuitive Approach to Faraday Effect

two eigenstates: left-handed circularly polarized state and right-handed circularly polarized state.

The former carries $+\hbar$ angular momentum, and the later $-\hbar$.

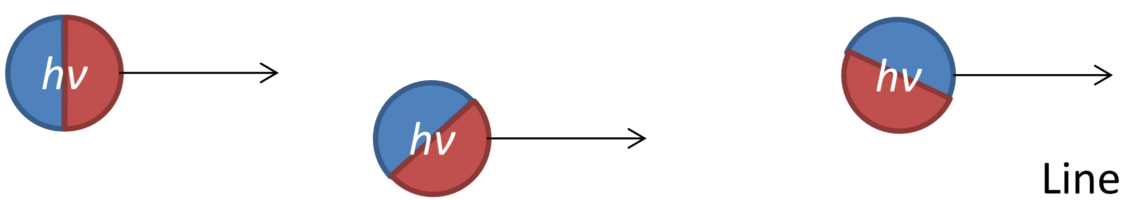


Left-handed circularly polarized light



Right-handed circularly polarized light

A linearly polarized light is the superposition of two circularly polarized light with the same amplitude, different relative phase will give different polarization direction



Linearly polarized light

An Intuitive Approach to Faraday Effect

- When interacting with the matter, the angular momentum carried by photons will be conveyed to the atom due to angular momentum conservation

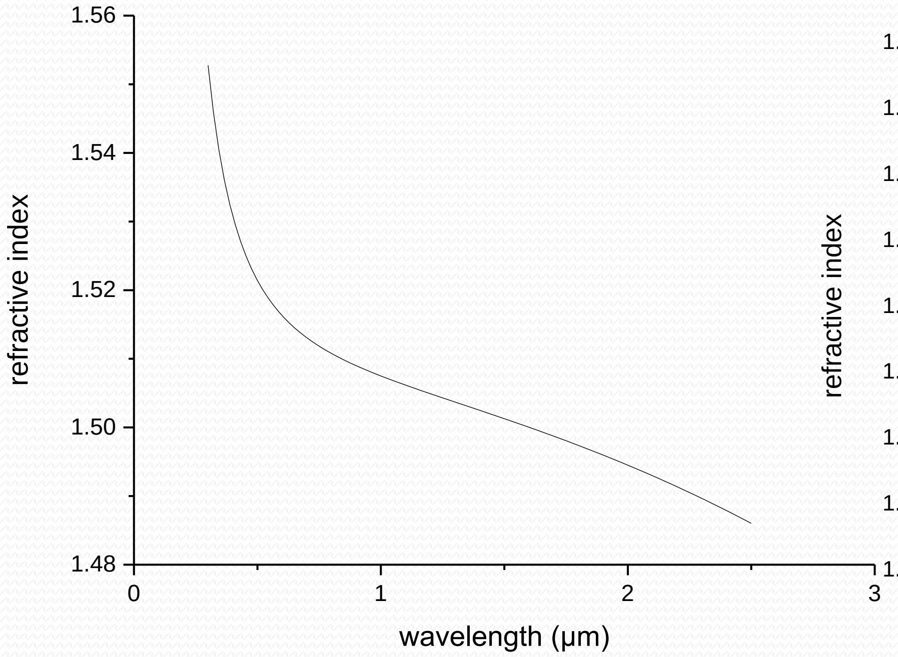
- When the atom is in the magnetic field, there will be an additional energy due to the angular momentum change

$$\Delta E = -\Delta \boldsymbol{\mu} \cdot \boldsymbol{B}_0 = -\frac{e}{2m} \Delta \boldsymbol{L} \cdot \boldsymbol{B}_0 = \mp \frac{eB_0\hbar}{2m}$$

- So a L-photon has effective energy of $h\nu - eB_0\hbar/2m$, and a R-photon has effective energy of $h\nu + eB_0\hbar/2m$

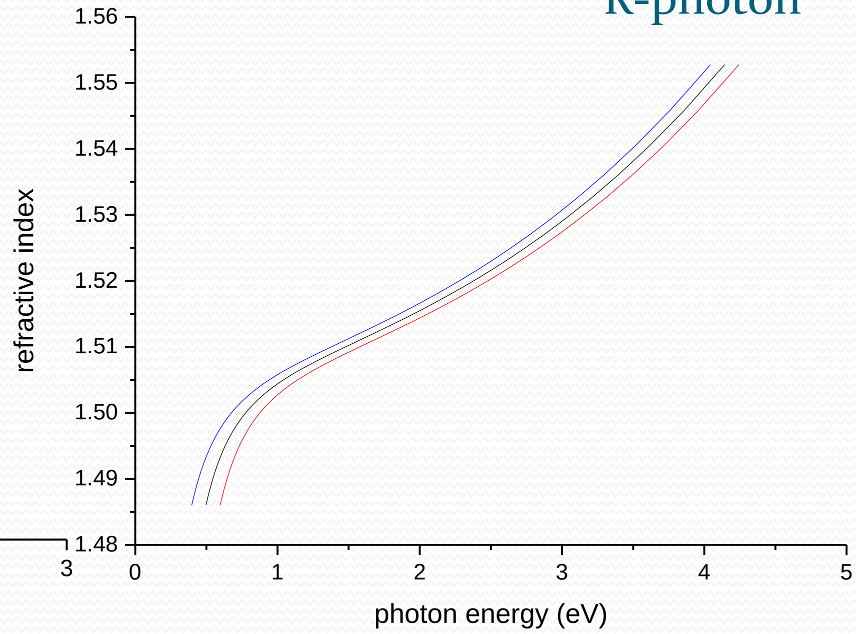
An Intuitive Approach to Faraday Effect

So the dispersion relation of L-photons and R-photons will no longer be the same. The dispersion will shift differently for two kinds of photons.



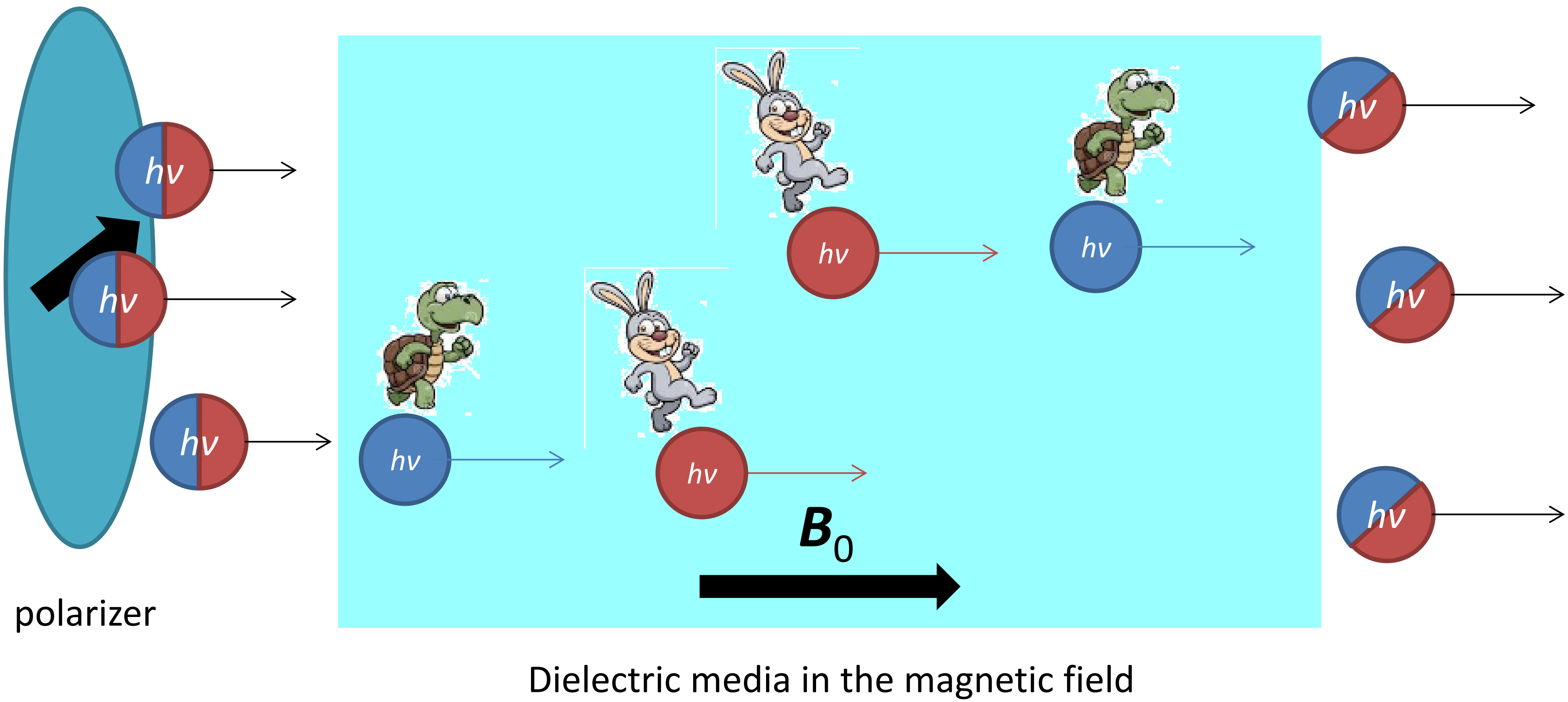
for BK7 glass, Data from http://refractiveindex.info/

L-photon
R-photon



effective dispersion relation
the shift of the curve is exaggerated

An Intuitive Approach to Faraday Effect



An Intuitive Approach to Faraday Effect

- For L-photon the effective dispersion relation

$$n_L(\omega) = n\left(\omega - \frac{eB_0}{2m}\right) = n(\omega) - \frac{eB_0}{2m} \cdot \frac{\mathrm{d}n}{\mathrm{d}\omega}$$

- The same way

$$n_R(\omega) = n\left(\omega + \frac{eB_0}{2m}\right) = n(\omega) + \frac{eB_0}{2m} \cdot \frac{\mathrm{d}n}{\mathrm{d}\omega}$$

- The rotation angle $\Delta\varphi$ of the polarization vector

$$\Delta\varphi = \frac{\Delta\theta}{2} = \omega \cdot D/c(n_R - n_L) = \frac{D}{2} \cdot \frac{B_0 e}{mc} \cdot \omega \frac{\mathrm{d}n}{\mathrm{d}\omega}$$

- Experimentalists denote it as $\Delta\varphi = V(\lambda)B_0 D$, where $V(\lambda)$is called the Verdet constant.
- This explanation is intuitive but somehow vague, a restrict proof is attached as appendix I, using the classical point of view.

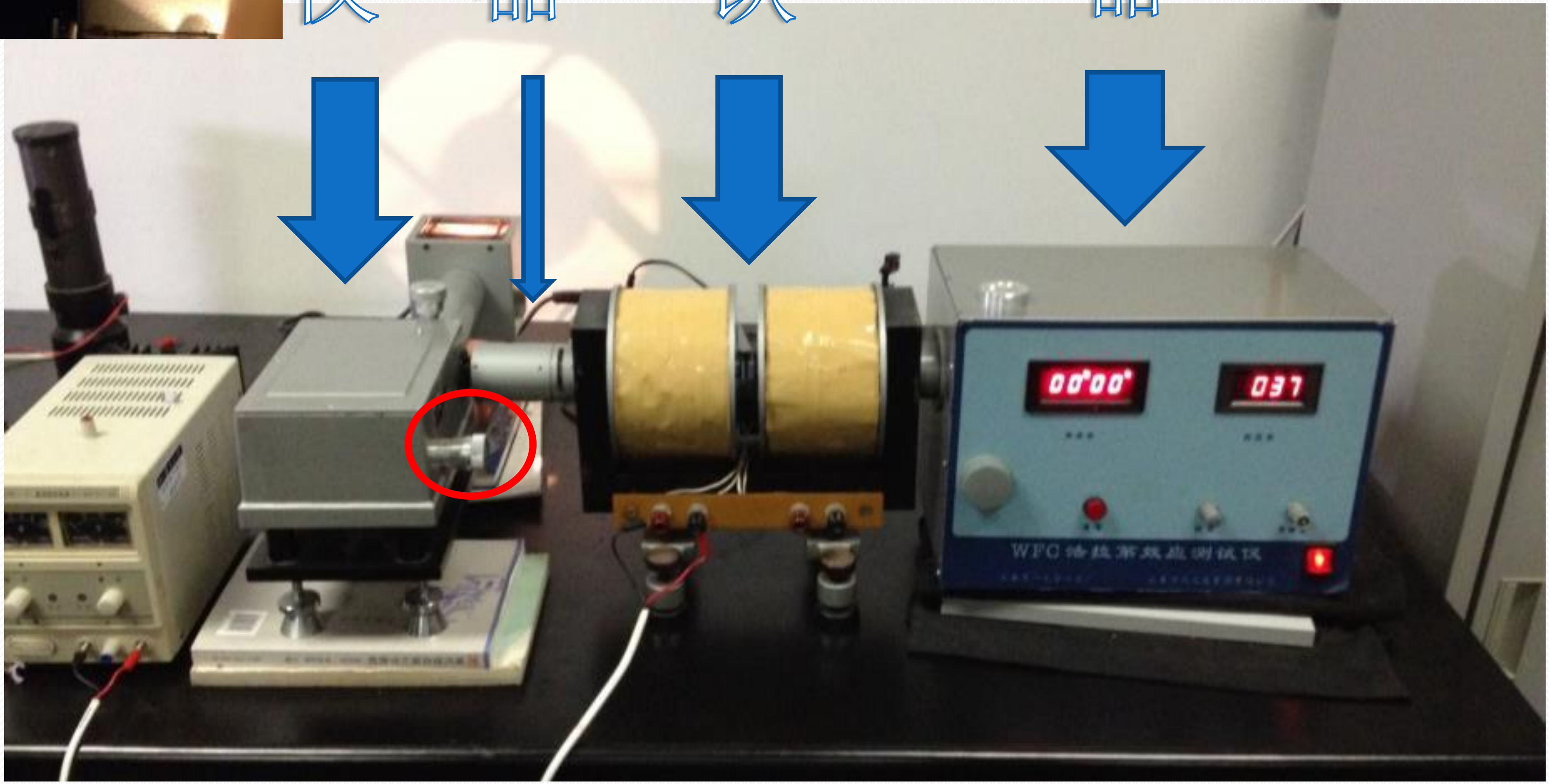
Experimental equipment

单色仪

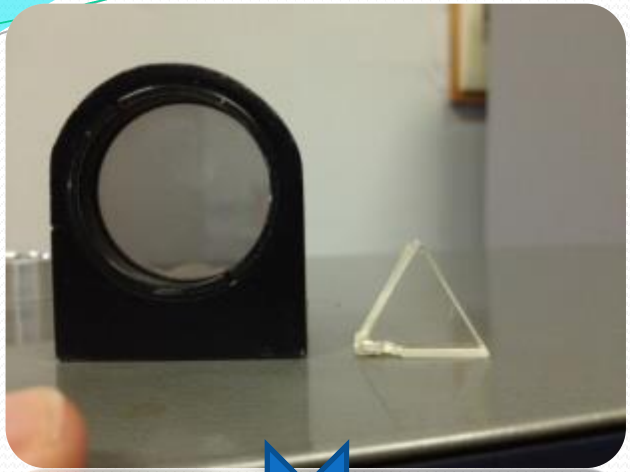
起偏器

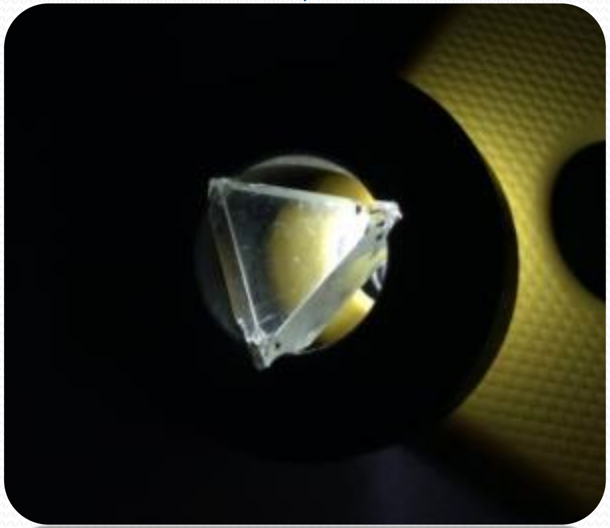
电磁铁

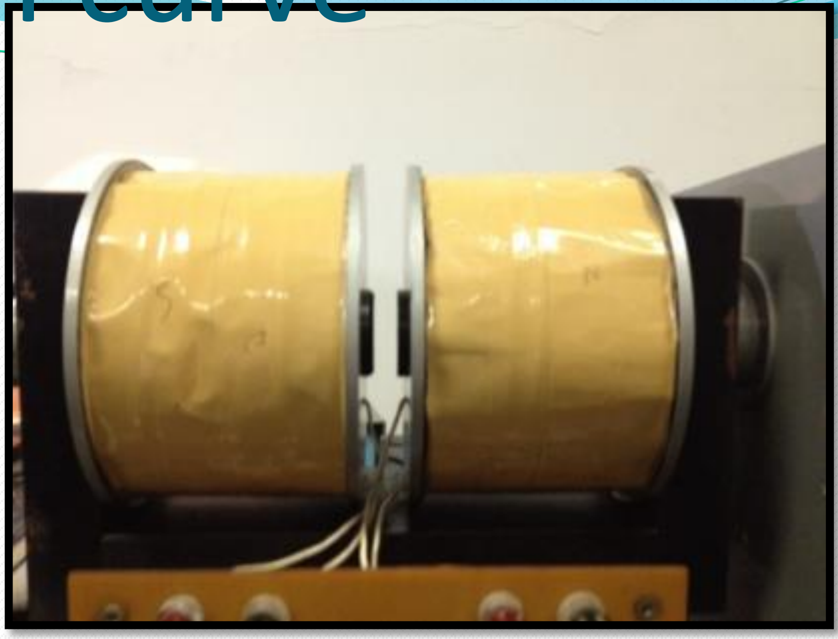
检偏器



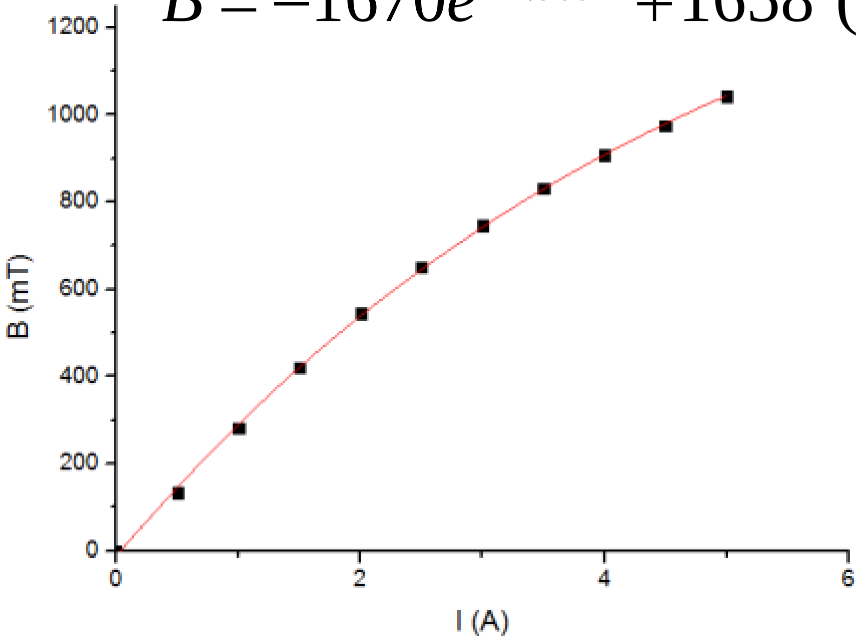
Electromagnet B-I curve





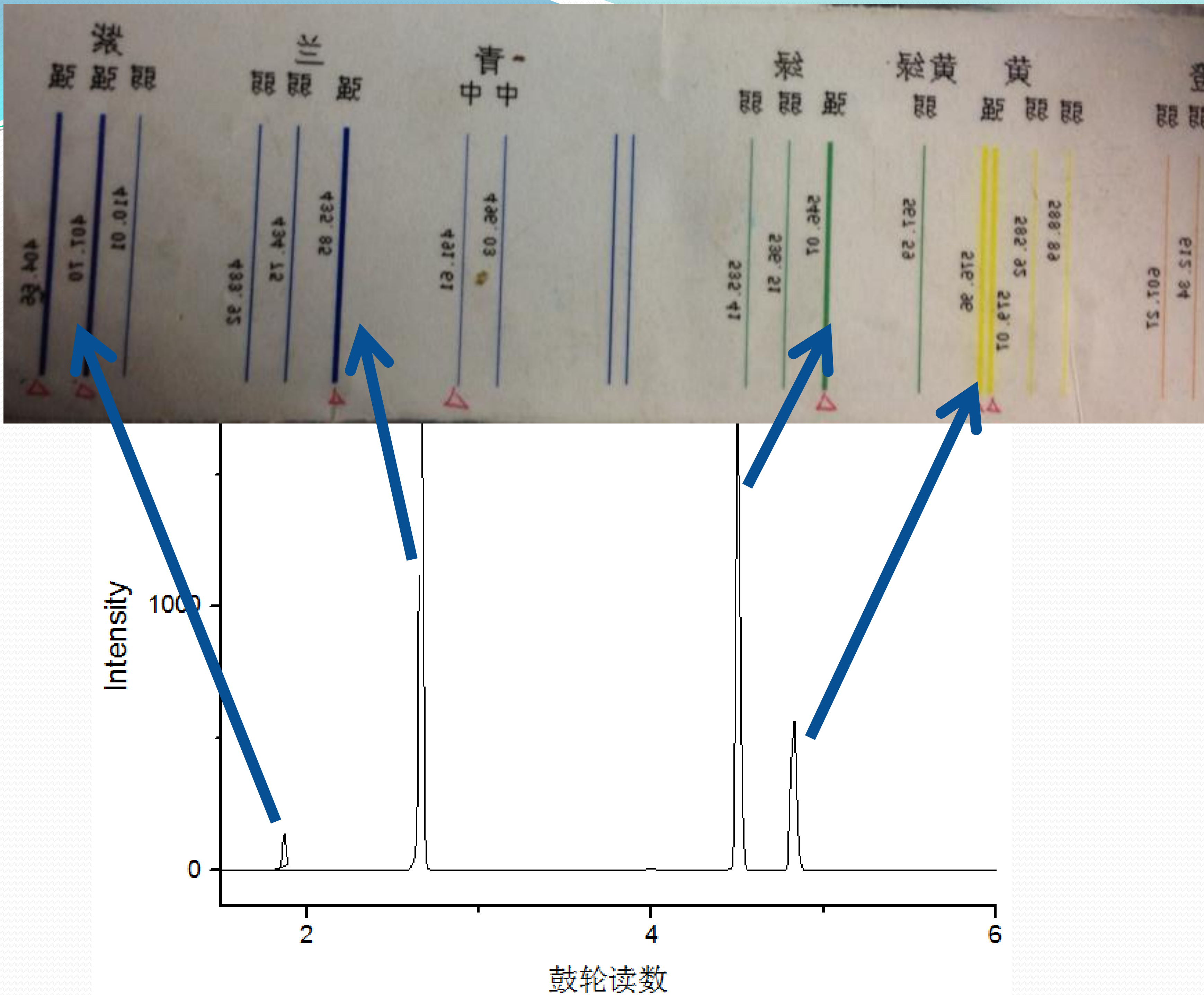


$$B = -1670e^{-\frac{I}{4.9897}} + 1658\ (mT)$$



Wavelength Calibration

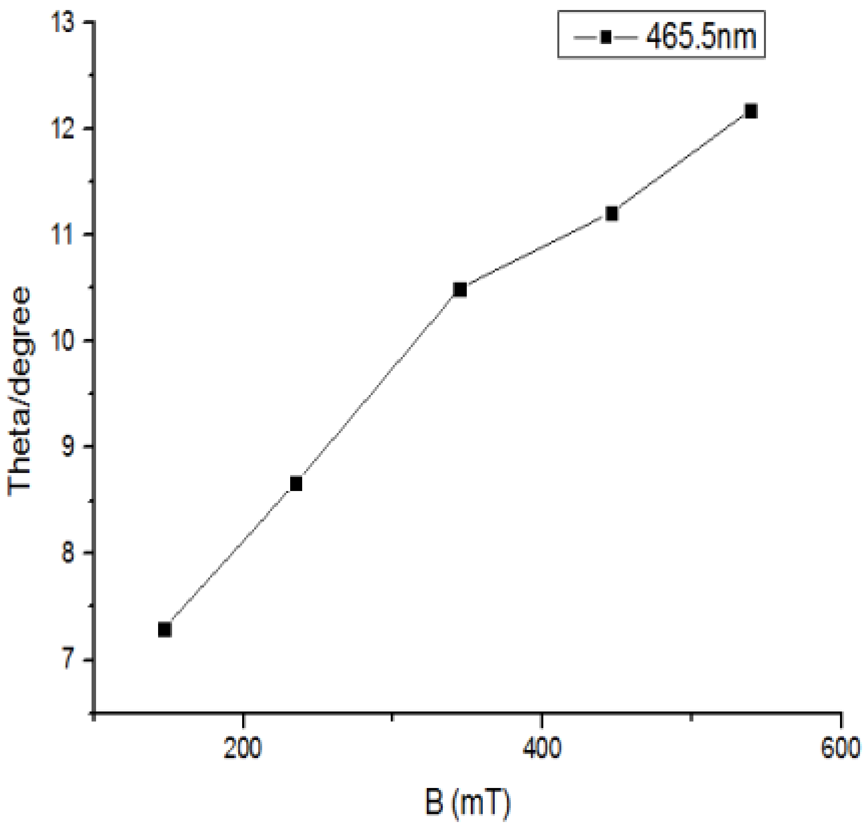
High Pressure
Mercury Lamp

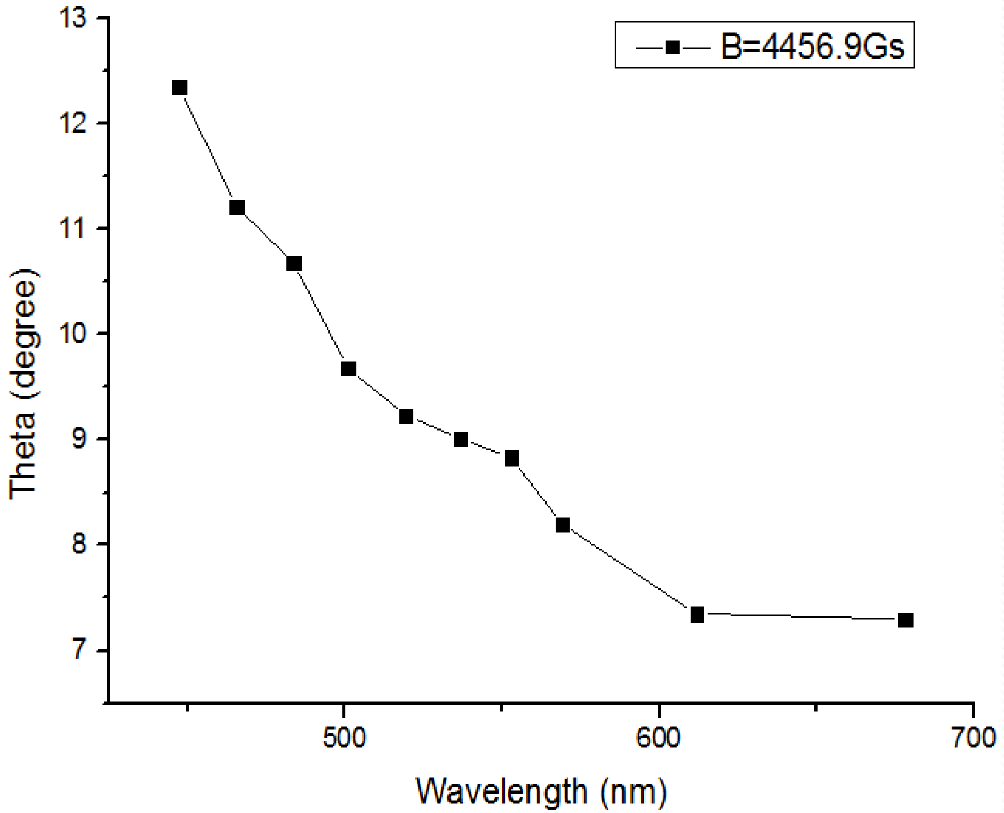


Intensity
1000
0
2
4
6
鼓轮读数

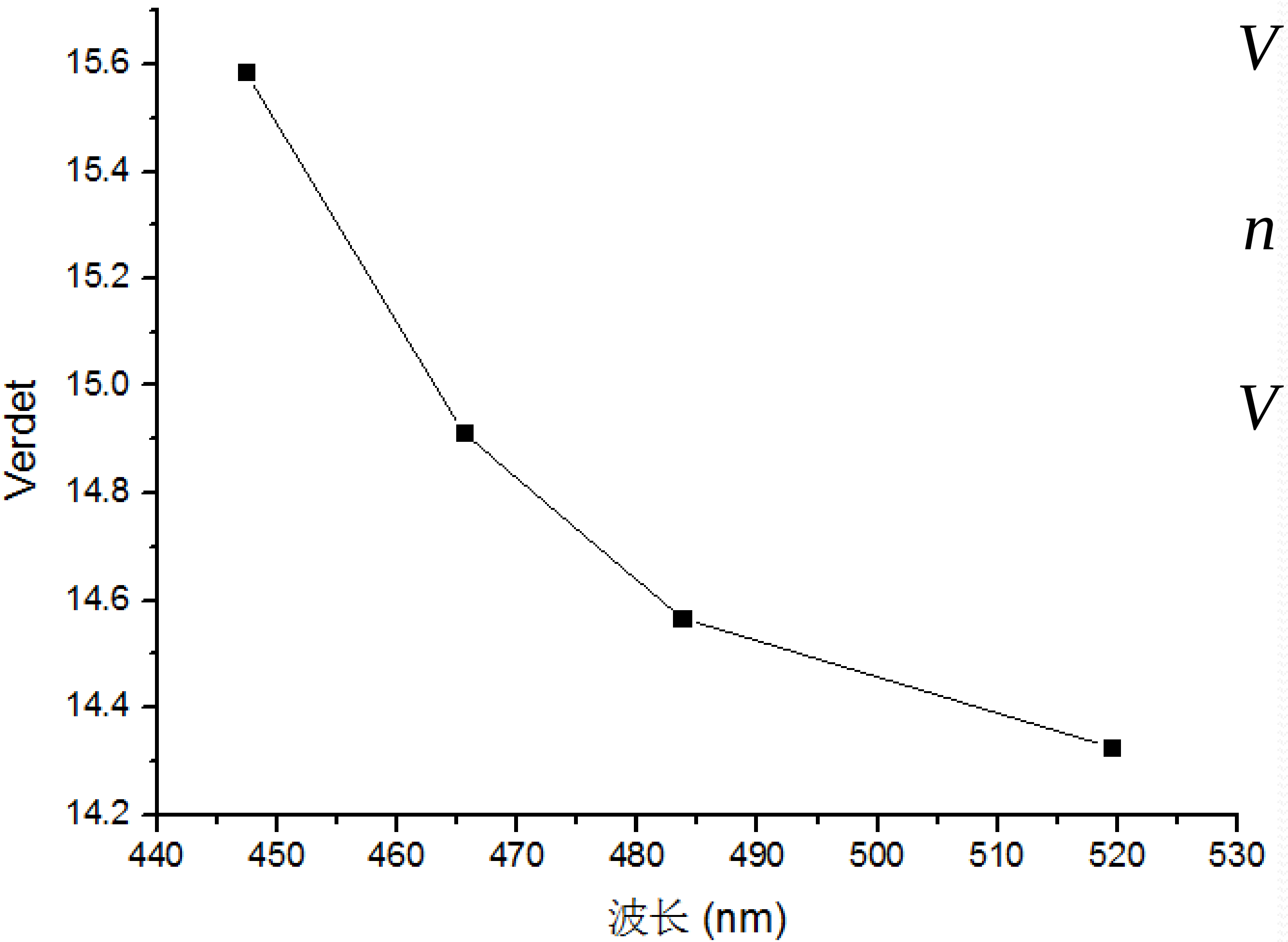
Θ~λ, Θ~B

$$\theta = V(\lambda)DB$$





V(λ)~ λ



$$V(\lambda) = -\frac{e}{2mc}\lambda\frac{dn}{d\lambda}$$

$$n = C + \frac{F}{\lambda^2} + \frac{G}{\lambda^4}$$

$$V(\lambda) \approx -\frac{e}{2mc}\frac{-2F}{\lambda^2}$$

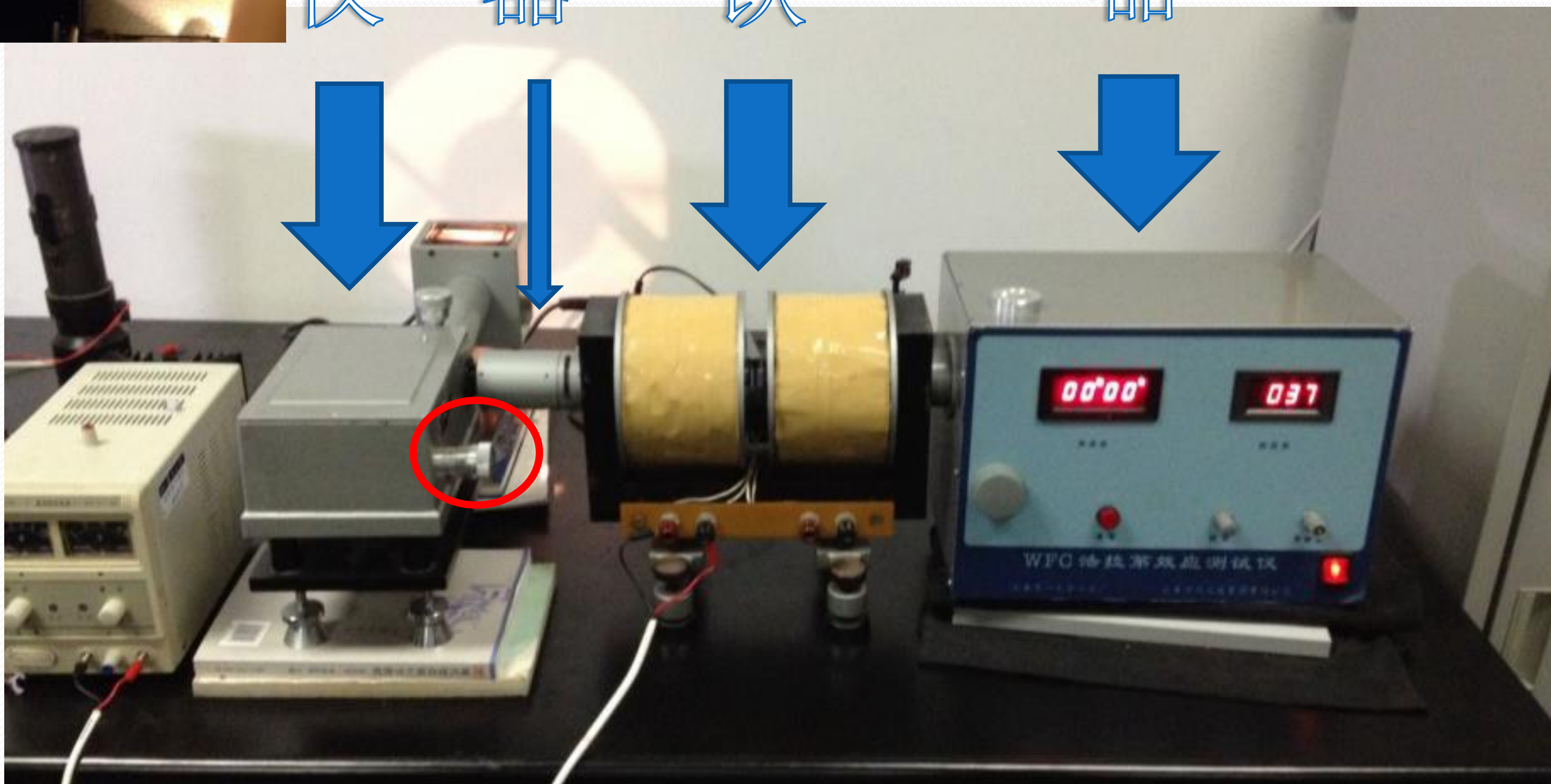
Problem encountering

单色仪

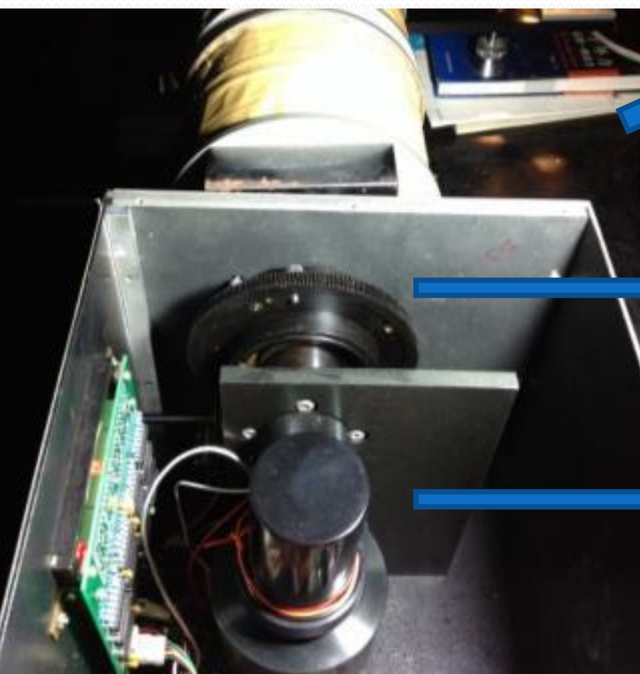
起偏器

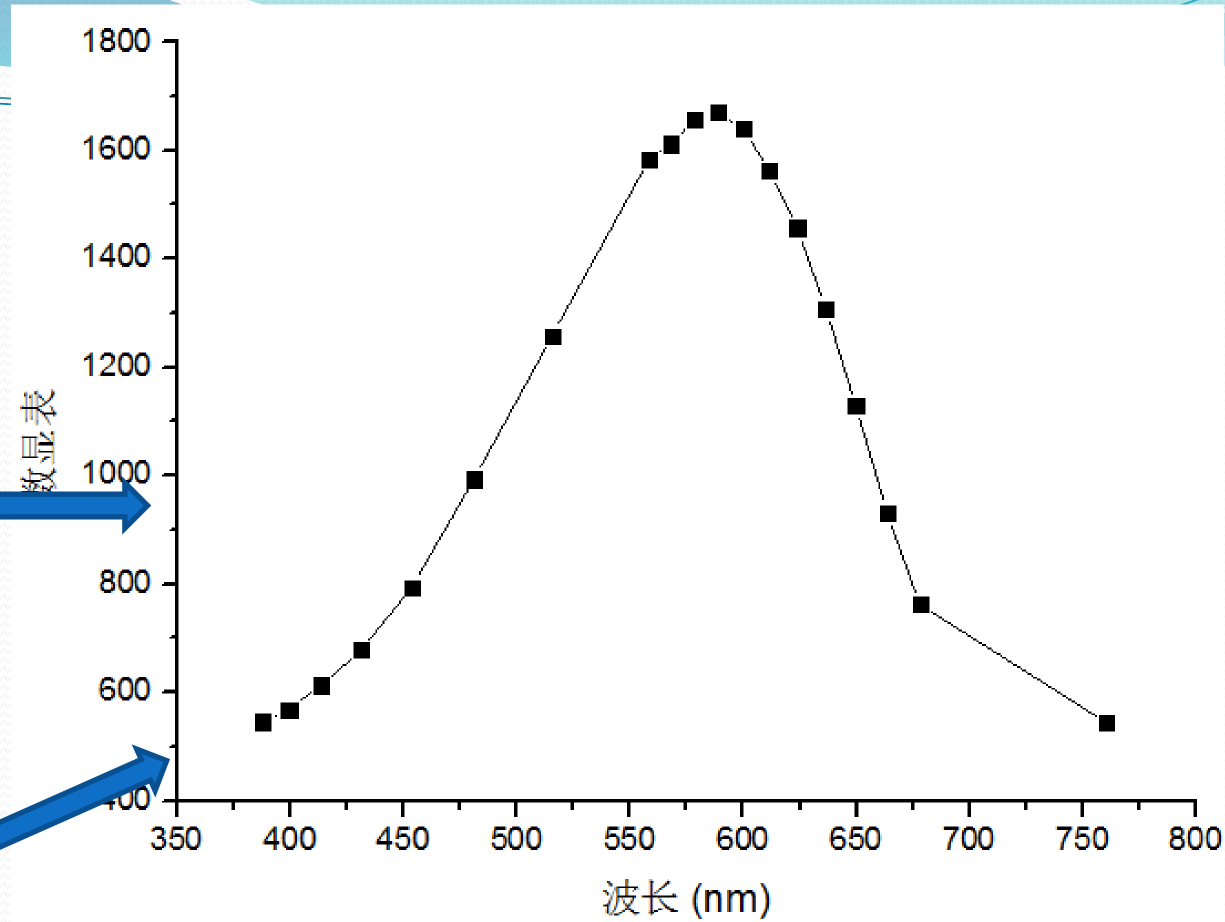
电磁铁

检偏器



Response



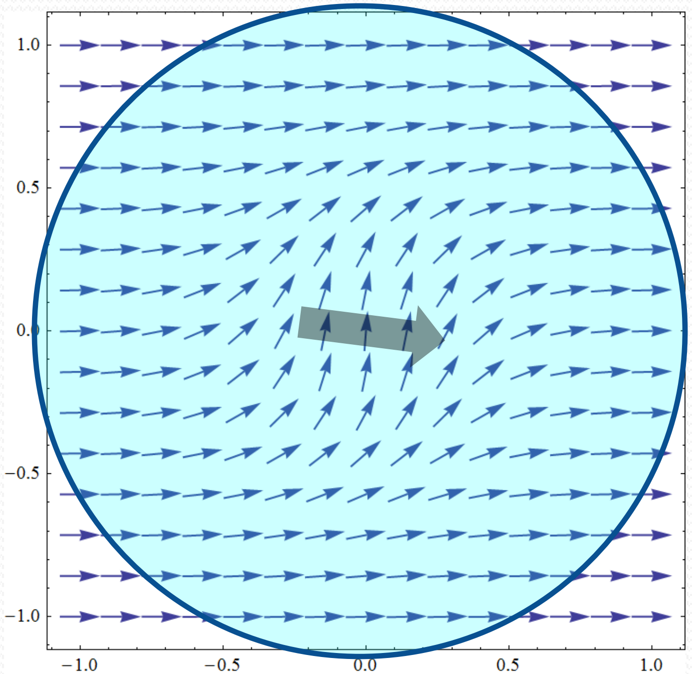


偏振片

光电转换器

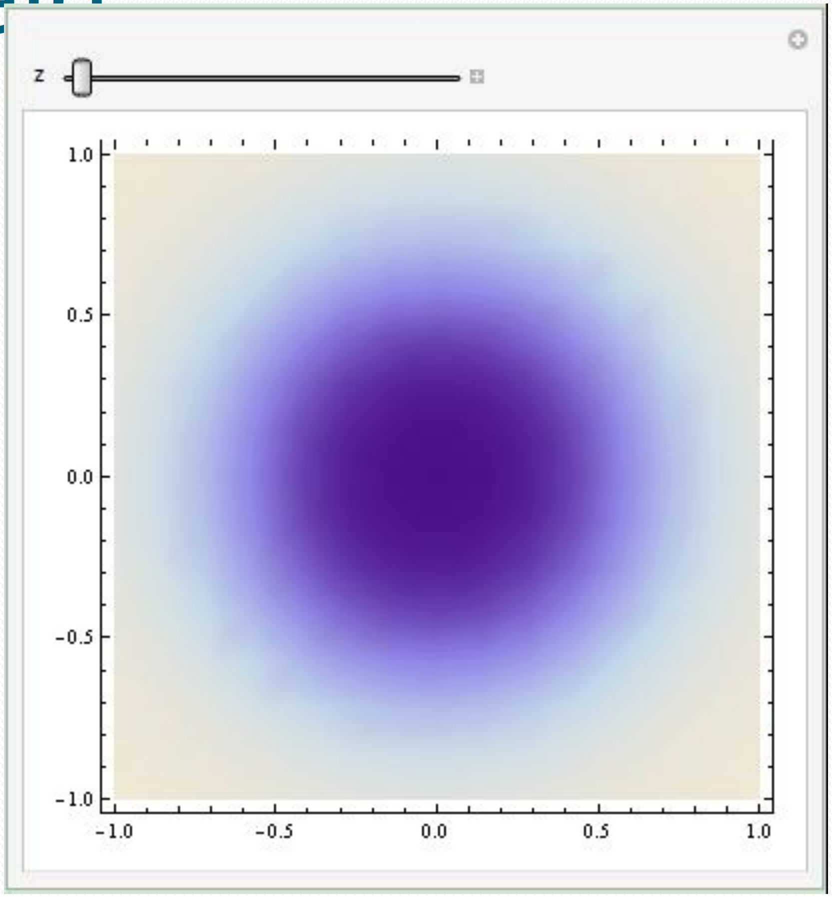
The effect of inhomogeneous *B* field

One problem has long bothered us: After turning on the magnetic field, we could not reduce the intensity to be zero by rotating the second polarizer. We tried hard to explain that and finally came up with a reasonable explanation: the ***B*** field is inhomogeneous

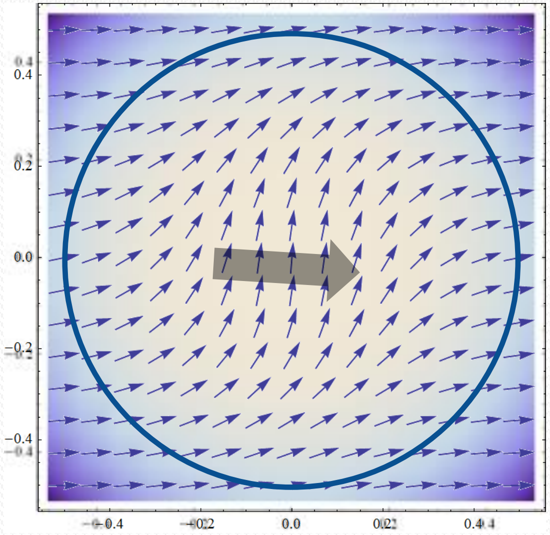


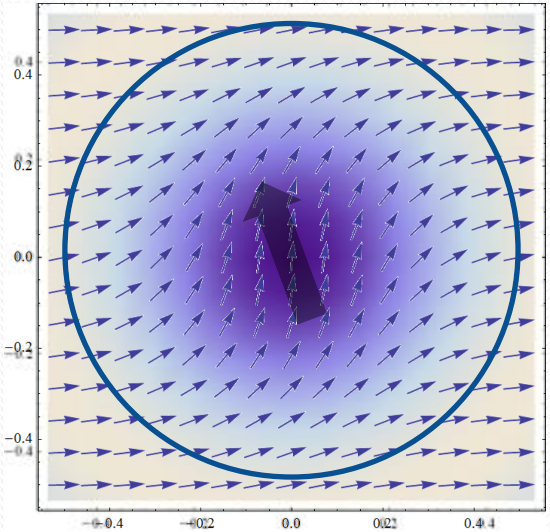
We suppose the magnetic field is a 2D Gaussian function centered at the origin. The polarization direction is shown in the left figure. We can not eliminate the light every where by rotating a polarizer

The effect of inhomogeneous *B* field



The light intensity change when the polarizer is rotating

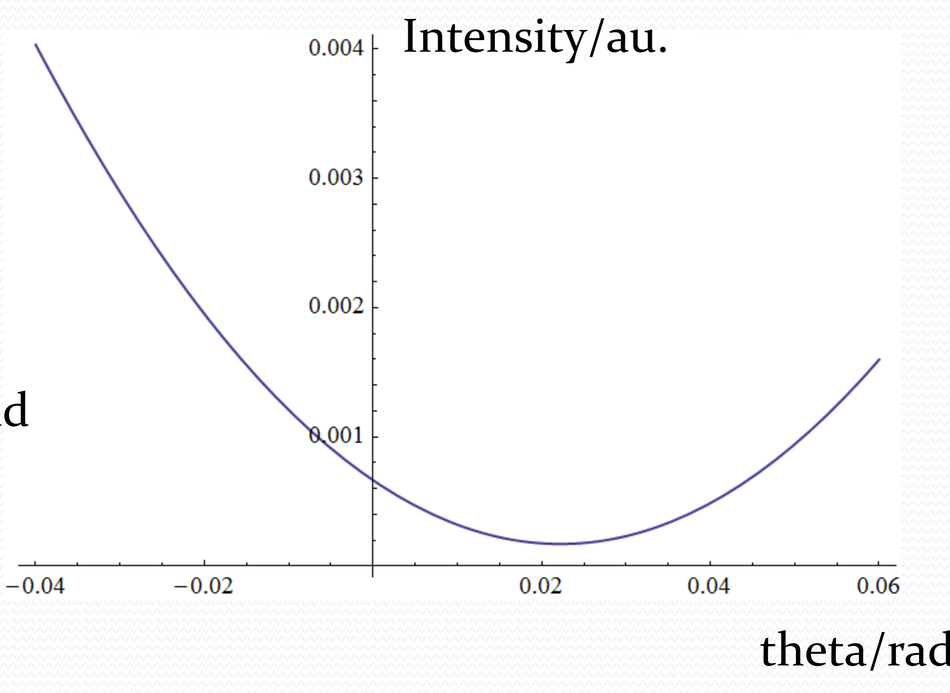




The effect of inhomogeneous ***B*** field

Now comes the question. We can only get the angle at which the gauge shows the minimal value. Will that be reasonable to use this angle as the rotating angle?

I have done some calculation
I=2A B_0=544mT D=10.1mm λ=550nm
Integrate over the circle centered at the origin with radius 5mm
The optical axis of the polarizer is nearly perpendicular to polarization direction of the light at origin. However we find the minimal value occurs at θ = 0.02rad instead of θ = 0.

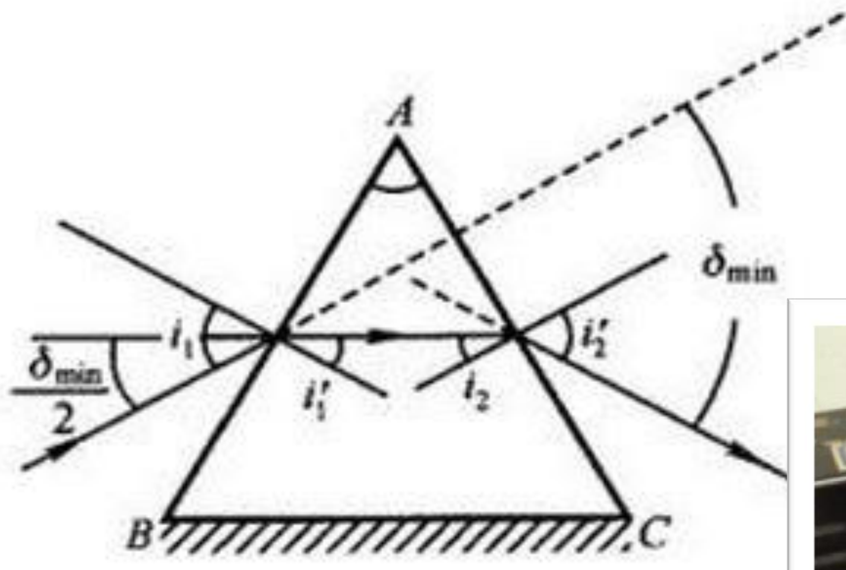


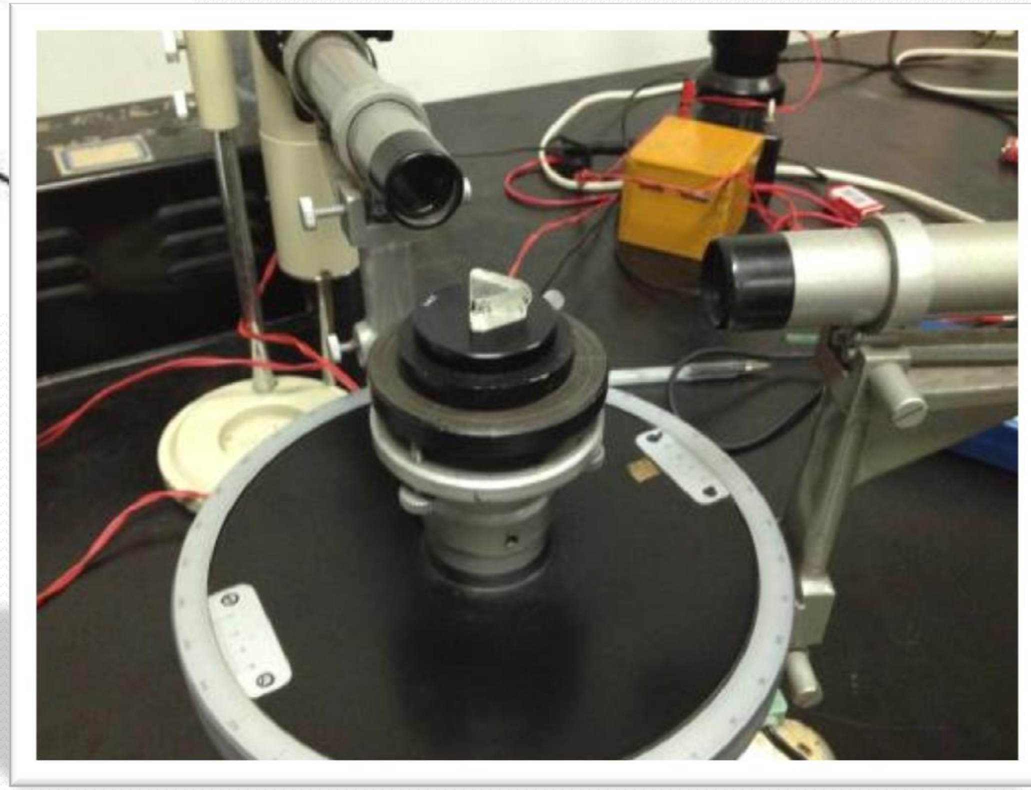
验证

$$\theta = \left(-\frac{e}{2mc}\lambda\frac{dn}{d\lambda}\right)DB$$

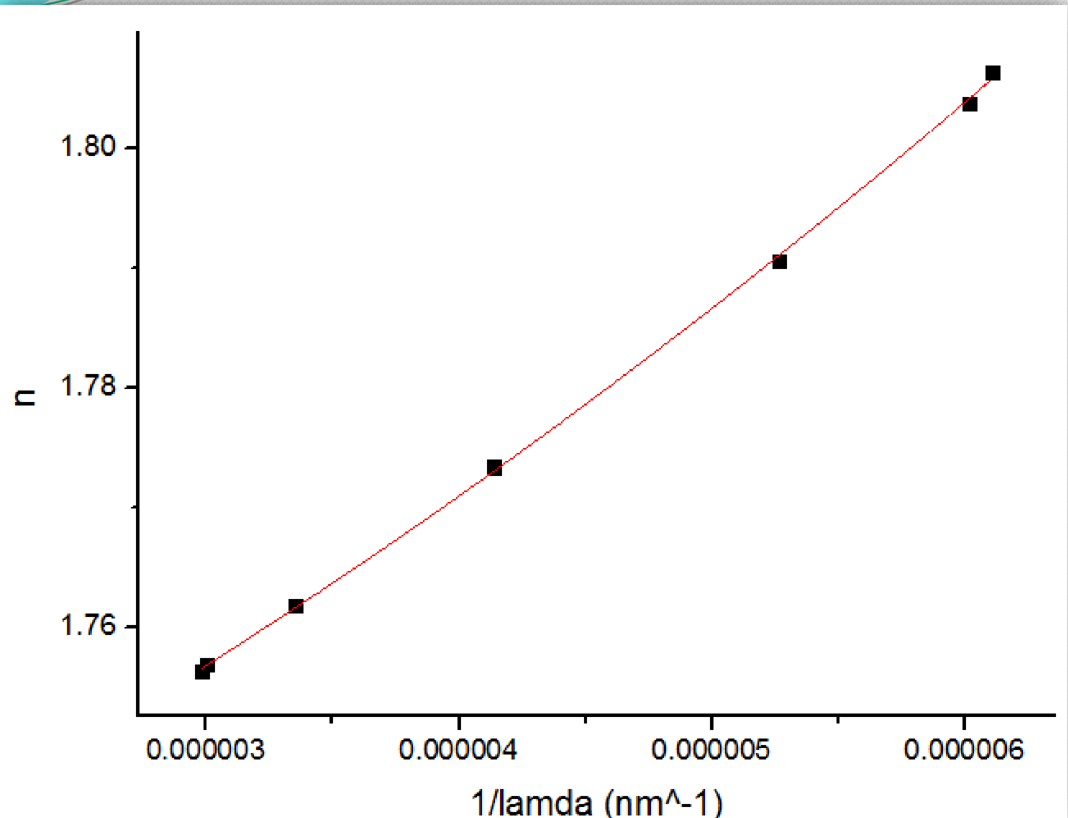
最小偏向角法测色散关系

$$n = \frac{\sin\frac{\delta(\lambda)+A}{2}}{\sin\frac{A}{2}}$$

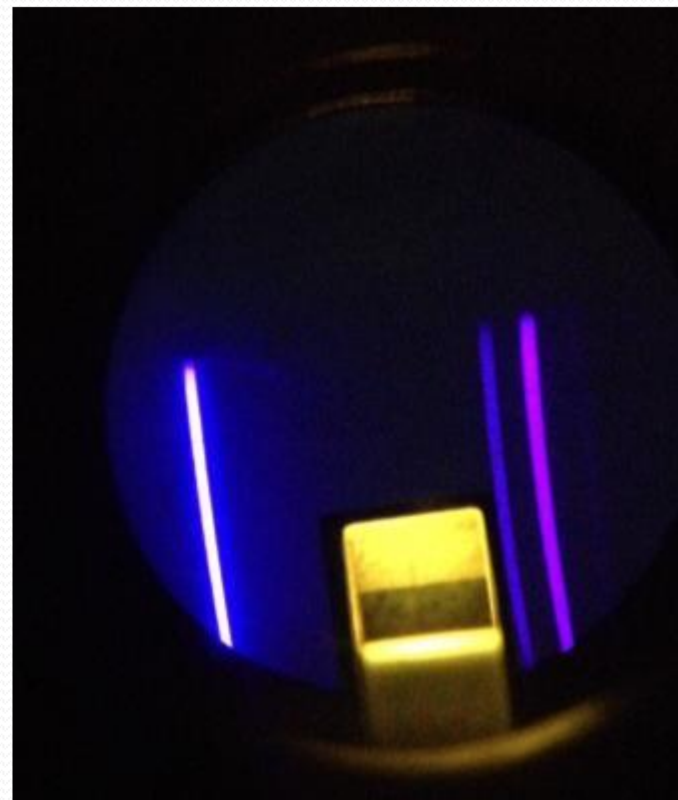




色散关系



$$n = 1.723 + \frac{8.852\times10^{-1}}{\lambda^2} + \frac{7.59\times10^{-28}}{\lambda^4}$$



e/m

$$\theta = (-\frac{e}{2mc}\lambda\frac{dn}{d\lambda})DB$$

$$\lambda = 483.68nm$$

$$B = 445.7mT$$

$$\frac{e}{m} = 1.894\times10^{11}c/kg$$

标准值 $\frac{e}{m} = 1.758\times10^{11}c/kg$

误差 7.7%

Conclusion:

$$\theta \propto B$$

The effect of inhomogeneous *B* field

$$n = 1.723 + \frac{8.852 \times 10^{-15}}{\lambda^2} + \frac{7.59 \times 10^{-28}}{\lambda^4}$$

$$V(\lambda) \sim \frac{1}{\lambda^2}$$

Appendix I: mathematical formulation of Faraday Effect

In general, the transverse electromagnetic plane wave propagating in vacuum along z direction can be written as

$$E_x = E_{x0}\cos(kz - \omega t + \varphi_x)$$
$$E_y = E_{y0}\cos(kz - \omega t + \varphi_y)$$

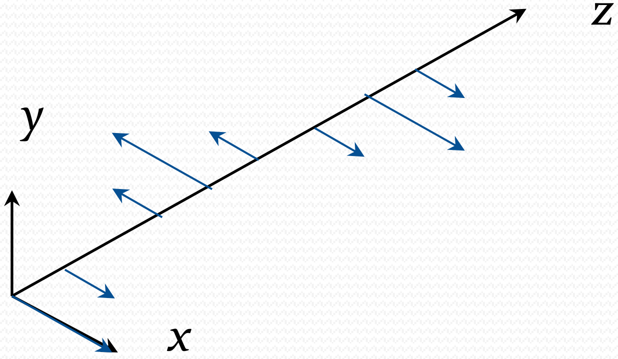
When using the complex amplitude , it can be written as

$$\widetilde{\boldsymbol{E}} = \begin{pmatrix} E_{x0}\mathrm{e}^{i\varphi_x} \\ E_{y0}\mathrm{e}^{i\varphi_y} \end{pmatrix} \mathrm{e}^{i(kz-\omega t)}$$

We can use two set of basis, say,

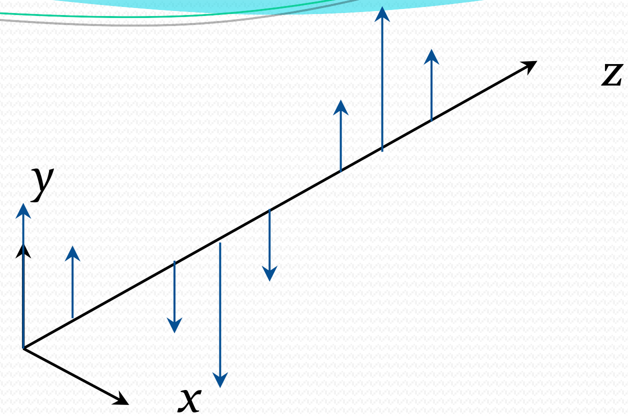
$$\boldsymbol{e}_1 = \binom{1}{0}, \boldsymbol{e}_2 = \binom{0}{1}; \boldsymbol{e'}_1 = \frac{1}{\sqrt{2}}\binom{1}{i}, \boldsymbol{e'}_2 = \frac{1}{\sqrt{2}}\binom{1}{-i}$$

$\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are related to linear polarization modes and $\boldsymbol{e'}_1$, $\boldsymbol{e'}_1$ are related to circular polarization modes.



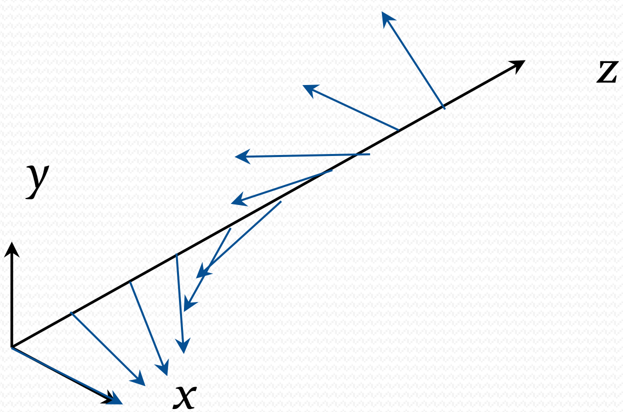
$\widetilde{\boldsymbol{E}} = E\boldsymbol{e}_1 \mathrm{e}^{i(kz-\omega t)}$

linearly polarized light



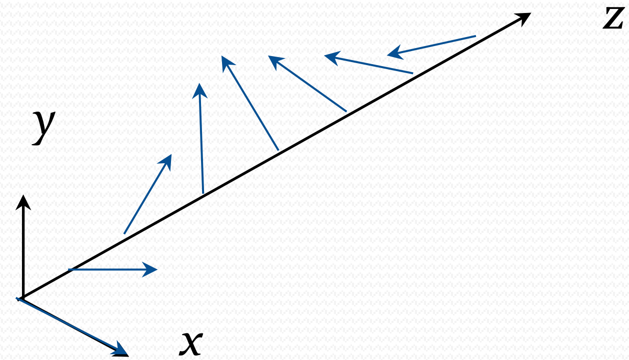
$\widetilde{\boldsymbol{E}} = E\boldsymbol{e}_2 \mathrm{e}^{i(kz-\omega t)}$

linearly polarized light



$\widetilde{\boldsymbol{E}} = E\boldsymbol{e}'_1 \mathrm{e}^{i(kz-\omega t)}$

left-hand circularly polarized light



$\widetilde{\boldsymbol{E}} = E\boldsymbol{e}'_2 \mathrm{e}^{i(kz-\omega t)}$

right-hand circularly polarized light

Appendix I

Decomposition of linearly polarized light

$$\boldsymbol{e}_1 = \frac{1}{\sqrt{2}}(\boldsymbol{e'}_1 + \boldsymbol{e'}_2), \boldsymbol{e}_2 = \frac{-i}{\sqrt{2}}(\boldsymbol{e'}_1 - \boldsymbol{e'}_2)$$

Now, we consider the propagation of electromagnetic wave in a dielectric, a uniform magnetic field $\boldsymbol{B}_0$ is parallel to the wave vector. Combine Newton's second law with Lorentz force formulation we get (assume $e<0$),

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

This equation governs the motion of the electrons in the dielectric. Suppose $\boldsymbol{B}_0$ is much larger than the magnet component of the electromagnetic wave, therefore $\boldsymbol{B} = \boldsymbol{B_0} = B_0\boldsymbol{z}$. And $\boldsymbol{E}$ is the electric component, $\boldsymbol{E} = \boldsymbol{E}_0\mathrm{e}^{-i\omega t}$.
Take the testing solution as the form of $\boldsymbol{v} = \boldsymbol{v}_0\mathrm{e}^{-i\omega t}$. This makes sense, because in the case of forced oscillations, electrons will the transient part of a solution will decay and leave the steady part as time goes on.

Substitute the testing solution into the equations

$$-i\omega v_{0x} = \frac{-e}{m} E_{0x} + \frac{-e}{m} v_{0y} B_0$$

$$-i\omega v_{0y} = \frac{-e}{m} E_{0y} - \frac{-e}{m} v_{0x} B_0$$

$$-i\omega v_{0z} = \frac{-e}{m} E_{0z}$$

Use the Cyclotron motion frequency $\omega_B = eB_0/m$ to simplify the equations

$$v_{0x} = \frac{-e}{m} \frac{1}{\omega^2 - {\omega_B}^2} \left(i\omega E_{0x} + \omega_B E_{0y} \right)$$

$$v_{0y} = \frac{-e}{m} \frac{1}{\omega^2 - {\omega_B}^2} \left(\omega_B E_{0x} - i\omega E_{0y} \right)$$

$$v_{0z} = -\frac{e}{im\omega} E_{0z}$$

Combine $\boldsymbol{j} = ne\boldsymbol{v}$ and $\boldsymbol{j} = \overleftrightarrow{\sigma}\boldsymbol{E}$, we can then get the conductivity tensor from $\boldsymbol{v} = \frac{1}{-ne}\overleftrightarrow{\sigma}\boldsymbol{E}$.

If we combine the Maxwell equation in the dielectric and $\boldsymbol{E} = \boldsymbol{E}_0 \mathrm{e}^{-i\omega t}$ we will get

$$\nabla \times \boldsymbol{H} = \left(\overleftrightarrow{\sigma} - i\omega\varepsilon_0 \overleftrightarrow{I}\right)\boldsymbol{E}$$

ε_0 is the intrinsic permittivity of the dielectric, $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}$

The effective permittivity tensor satisfies $\nabla \times \boldsymbol{H} = -i\omega\overleftrightarrow{\varepsilon}\boldsymbol{E}$

Therefore, $\overleftrightarrow{\varepsilon} = \varepsilon_0 \overleftrightarrow{I} - \frac{1}{i\omega}\overleftrightarrow{\sigma}$

We can denote it as $\overleftrightarrow{\varepsilon} = \varepsilon_0 \overleftrightarrow{\varepsilon_r} = \varepsilon_0 \begin{pmatrix} \varepsilon_1 & i\varepsilon_2 & 0 \\ -i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$

where $\varepsilon_1 = 1 - \frac{{\omega_p}^2}{\omega^2 - {\omega_B}^2}$, $\varepsilon_2 = \frac{{\omega_p}^2 \omega_B}{(\omega^2 - {\omega_B}^2)\omega}$, $\varepsilon_3 = 1 - \frac{{\omega_p}^2}{\omega}$,and

${\omega_p}^2 = \frac{ne^2}{m\varepsilon_0}$. We assume $\omega > \omega_p \gg \omega_B$, so that $\varepsilon_1 \gg \varepsilon_2$.

We see the magnetic field contributes to the off-diagonal element of the permittivity tensor, and it is pure imaginary.

Having got the permittivity tensor , let us find the plane wave solution of the Maxwell equation. We are interested in the plane wave propagating along z direction$\boldsymbol{k} = k\hat{z}$, i.e. the direction of the external magnetic field. Substitute $\boldsymbol{E} = \boldsymbol{E}_0 \mathrm{e}^{i(kz-\omega t)}$ and $\boldsymbol{H} = \boldsymbol{H}_0 \mathrm{e}^{i(kz-\omega t)}$into the Maxwell equation

$$\begin{cases} \nabla \cdot \boldsymbol{D} = 0 \\ \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{H} = \dfrac{\partial \boldsymbol{D}}{\partial t} \end{cases}$$

We get

$$\begin{cases} \boldsymbol{k} \cdot (\overleftrightarrow{\varepsilon} \cdot \boldsymbol{E}_0) = 0 \\ \boldsymbol{k} \times \boldsymbol{E}_0 = \omega \mu_0 \boldsymbol{H}_0 \\ \boldsymbol{k} \cdot \boldsymbol{H}_0 = 0 \\ \boldsymbol{k} \times \boldsymbol{H}_0 = -\omega \overleftrightarrow{\varepsilon} \cdot \boldsymbol{E}_0 \end{cases}$$

From the first equation we know that $E_z = 0$, therefore the electromagnetic wave is transverse.

Combining the second and the fourth equations, as well as the condition that $\boldsymbol{k} = k\hat{z}$ we know that

$$k^2 \boldsymbol{E}_0 = \omega^2 \mu_0 \varepsilon_0 \overleftrightarrow{\varepsilon_r} \cdot \boldsymbol{E}_0$$

Noting the wave vector in vacuum is $k_0 = \omega/c = \omega\sqrt{\mu_0 \varepsilon_0}$

$$({k_0}^2 \overleftrightarrow{\varepsilon_r} - k^2 \overleftrightarrow{I})\boldsymbol{E}_0 = 0$$

$k^2$is the eigenvalue of ${k_0}^2 \overleftrightarrow{\varepsilon_r}$, solving the secular equation brings us with

$$k_1 = k_0\sqrt{\varepsilon_1 + \varepsilon_2}, k_2 = k_0\sqrt{\varepsilon_1 - \varepsilon_2}, k_3 = k_0\sqrt{\varepsilon_3}$$

the corresponding eigenvector should be

$$\boldsymbol{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \boldsymbol{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The z component of the electric field is 0, so $\boldsymbol{E}_0 = E_1 \boldsymbol{e}_1 + E_2 \boldsymbol{e}_2$, these two components are just right-handed polarized light and the left-handed polarized light. They have different phase velocity in the media.

Before entering the media, a linearly polarized light can also be viewed as the superposition of two circularly polarized light

$$\boldsymbol{E} = E_0 \boldsymbol{e}_x e^{i(k_0 z - \omega t)} = E_0/\sqrt{2}\,(\boldsymbol{e}_1 + \boldsymbol{e}_2) e^{i(k_0 z - \omega t)}$$

After entering the media, the two components of circular polarized light begin to propagate in different phase velocity.

$$\boldsymbol{E} = E_0/\sqrt{2}\left(\boldsymbol{e}_1 e^{ik_1 z} + \boldsymbol{e}_2 e^{ik_2 z}\right) e^{-i\omega t}$$

$$= E_0/\sqrt{2}\left(\boldsymbol{e}_1 e^{-i\Delta kz/2} + \boldsymbol{e}_2 e^{i\Delta kz/2}\right) e^{i(\bar{k}z - \omega t)}$$

$$= E_0/2\left(\boldsymbol{e}_x\left(e^{-\frac{i\Delta kz}{2}} + e^{\frac{i\Delta kz}{2}}\right) + \boldsymbol{e}_y\left(e^{-\frac{i\Delta kz}{2}} - e^{\frac{i\Delta kz}{2}}\right)\right) e^{i(\bar{k}z - \omega t)}$$

$$= E_0\left(\boldsymbol{e}_x \cos(\Delta kz/2) + \boldsymbol{e}_y \sin(\Delta kz/2)\right) e^{i(\bar{k}z - \omega t)}$$

Where $\Delta k = k_1 - k_2$, $\bar{k} = (k_1 + k_2)/2$

The last equation indicates that for each point in the media, the electromagnetic wave vector is rotating in an angular velocity of ω.

Compare the point $z = 0$ and $z = D$, we get the angle rotated

$$\Delta\varphi = \Delta kD/2 = k_0 D\left(\sqrt{\varepsilon_1 + \varepsilon_2} - \sqrt{\varepsilon_1 - \varepsilon_2}\right)/2 = k_0 D \varepsilon_2/\sqrt{\varepsilon_1}$$

Now let's prove it's consistent with the result we got through the simple picture at the beginning

What we get at the beginning was $\Delta\varphi = \frac{D}{2} \cdot \frac{B_0 e}{mc} \cdot \omega \frac{\mathrm{d}n}{\mathrm{d}\omega}$

Noting that $n = \frac{c}{v} = c\sqrt{\varepsilon\mu} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}} = \sqrt{\varepsilon_r\mu_r} \approx \sqrt{\varepsilon_r}$

$$\frac{\mathrm{d}n}{\mathrm{d}\omega} = \frac{\mathrm{d}\sqrt{\varepsilon_r}}{\mathrm{d}\omega} = \frac{1}{2\sqrt{\varepsilon_r}} \frac{\mathrm{d}\varepsilon_r}{\mathrm{d}\omega}$$

Substitute $\varepsilon_r = 1 - \frac{{\omega_p}^2}{\omega^2 - {\omega_B}^2}$ into the equation we get

$$\frac{\mathrm{d}n}{\mathrm{d}\omega} = \frac{\mathrm{d}\sqrt{\varepsilon_r}}{\mathrm{d}\omega} = \frac{1}{\sqrt{\varepsilon_r}} \frac{{\omega_p}^2\omega}{(\omega^2 - {\omega_B}^2)^2}$$

So

$$\Delta\varphi = \frac{D}{2} \cdot \frac{B_0 e}{mc} \cdot \frac{1}{\sqrt{\varepsilon_r}} \frac{{\omega_p}^2\omega^2}{(\omega^2 - {\omega_B}^2)^2} \approx \frac{D}{2} \cdot \frac{B_0 e}{mc} \cdot \frac{1}{\sqrt{\varepsilon_r}} \cdot \frac{{\omega_p}^2}{\omega^2}$$

On the other hand, we have just get

$$\Delta\varphi = \Delta k D/2$$

Δk is the difference between $k_1 = k_0\sqrt{\varepsilon_1 + \varepsilon_2}$ and $k_2 = k_0\sqrt{\varepsilon_1 - \varepsilon_2}$

Expand the function $k(\omega) = k(\varepsilon_r) = k_0\sqrt{\varepsilon_r}$ near $\varepsilon_r = \varepsilon_1$

$$\Delta k \approx \left.\frac{\mathrm{d}k}{\mathrm{d}\varepsilon_r}\right|_{\varepsilon_r=\varepsilon_1} \cdot \Delta\varepsilon_r = \frac{k_0}{2\sqrt{\varepsilon_r}} \cdot 2\varepsilon_2$$

Substitute $\varepsilon_2 = \frac{{\omega_p}^2\omega_B}{(\omega^2-{\omega_B}^2)\omega}$ and $\omega_B = \frac{eB_0}{m}$ into the equation

$$\Delta\varphi = \frac{\Delta k D}{2} = \frac{k_0 D}{2\sqrt{\varepsilon_r}} \cdot \frac{eB_0}{m} \cdot \frac{{\omega_p}^2}{(\omega^2 - {\omega_B}^2)\omega}$$

$$\approx \frac{D}{2\sqrt{\varepsilon_r}} \cdot \frac{eB_0}{mc} \cdot \frac{{\omega_p}^2}{\omega^2}$$

The two pictures give the same result.

Reference: Prof. Lei Zhou’s lecture notes

Appendix II: Refractive Index in the view of Lorentz Model

In the view of classical physics, Lorentz assumes the nucleus of the atom is much more massive than the electron, then electrons can treated as connected to an infinite mass through a spring. In the external field of the light shed on the atom, the motion of the electron is described an forced oscillator

$$m\ddot{r} + g\dot{r} + kr = -eE_0 \mathrm{e}^{-i\omega t}$$

We can simplify it as

$$\ddot{r} + \gamma\dot{r} + {\omega_0}^2 r = -\frac{eE_0}{m}\mathrm{e}^{-i\omega t}$$

Where $\omega_0 = \sqrt{\frac{k}{m}}$ is the intrinsic angular frequency. $\gamma = \frac{g}{m}$ is the damping constant, the steady solution of the equation is

$$r = -\frac{eE_0}{m}\frac{1}{\omega^2 - {\omega_0}^2 + i\gamma\omega}\mathrm{e}^{-i\omega t}$$

The displacement of the electron will cause the polarization of the dielectric

$$\tilde{P} = -NZe\tilde{r}$$

$\tilde{P}$ and $\tilde{r}$ contain the phase factor so we use tilde t denote them. The complex permittivity is given by

$$\tilde{\varepsilon} = 1 + \chi_e = 1 + \frac{\tilde{P}}{\tilde{r}} = 1 - \frac{NZe^2}{\varepsilon_0 m} \frac{1}{\omega^2 - {\omega_0}^2 + i\gamma\omega}$$

This is the case for unique intrinsic angular frequency. Generally the atom has several kinds of oscillators with intrinsic angular frequency of $\omega_1, \omega_2, \omega_3$ … and the corresponding damping constant are $\gamma_1, \gamma_2, \gamma_3$ …the number of the oscillators aref_1, f_2, f_3 …

$$\tilde{\varepsilon} = 1 - \frac{Ne^2}{\varepsilon_0 m} \sum_j \frac{f_j}{\omega^2 - {\omega_j}^2 + i\gamma_j\omega_j}$$

Where $\Sigma f_j = Z$. the complex refractive index which describes both refraction and absorption can be get through $\tilde{n} = \sqrt{\tilde{\varepsilon}}$

$$\tilde{n} = \sqrt{\tilde{\varepsilon}} \approx 1 - \frac{Ne^2}{2\varepsilon_0 m} \sum_j \frac{f_j}{\omega^2 - \omega_j{}^2 + i\gamma_j\omega_j}$$

Refractive Index

$$n(\omega) = \mathrm{Re}(\tilde{n}) = 1 - \frac{Ne^2}{2\varepsilon_0 m} \sum_j \frac{f_j(\omega^2 - \omega_{0j}{}^2)}{(\omega^2 - \omega_j{}^2)^2 + \gamma_j\omega_j{}^2}$$

Use the wavelength $\lambda = 2\pi c/\omega$ and define $\lambda_j = 2\pi c/\omega_j$, n can be written as

$$n(\lambda) = 1 + \frac{Ne^2}{2\varepsilon_0 m} \sum_j \frac{f_j(\lambda^2 - \lambda_j{}^2)\lambda^2\lambda_j{}^2}{(2\pi c)^2(\lambda^2 - \lambda_j{}^2)^2 + \gamma_j{}^2\lambda^2\lambda_j{}^4}$$

At normal dispersion region $\lambda_j{}^4$ term can be omitted in the denominator

$$1 + \frac{Ne^2}{2\varepsilon_0 m} \sum_j \frac{a_j\lambda^2}{\lambda^2 - \lambda_j{}^2}$$

Where $a_j = Ne^2\lambda^2\lambda_j{}^2/2\varepsilon_0 m(2\pi c)^2$ is a constant

$$\tilde{n} = \sqrt{\tilde{\varepsilon}} \approx 1 - \frac{Ne^2}{2\varepsilon_0 m} \sum_j \frac{f_j}{\omega^2 - {\omega_j}^2 + i\gamma_j \omega_j}$$

Refractive Index

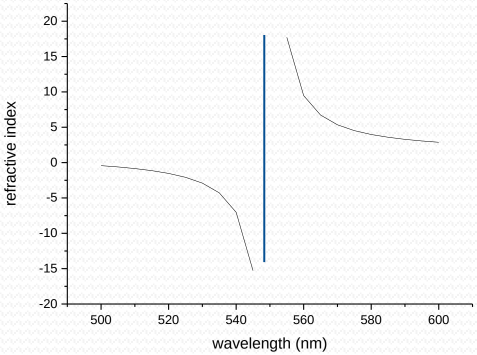
$$n(\omega) = \mathrm{Re}(\tilde{n}) = 1 - \frac{Ne^2}{2\varepsilon_0 m} \sum_j \frac{f_j(\omega^2 - {\omega_{0j}}^2)}{(\omega^2 - {\omega_j}^2)^2 + \gamma_j {\omega_j}^2}$$

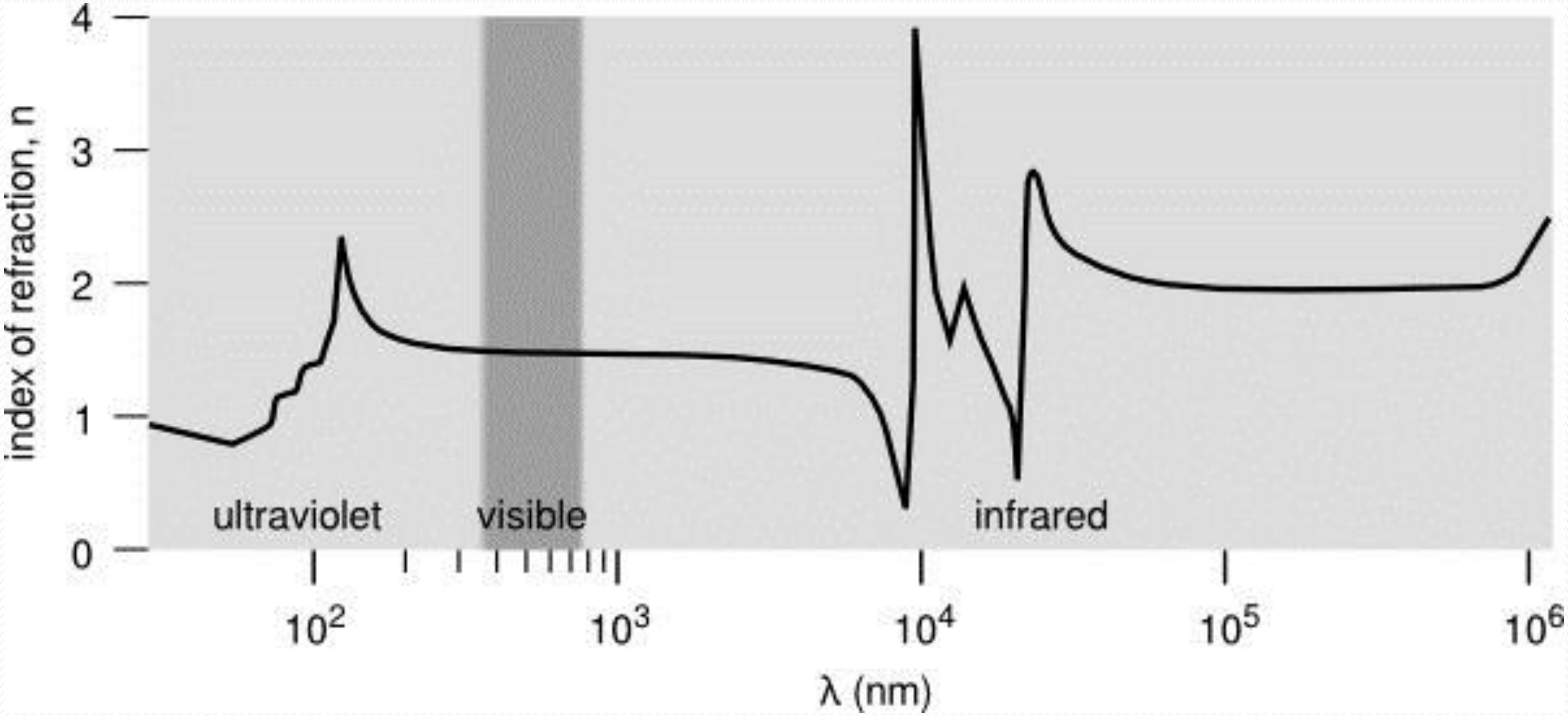
Use the wavelength $\lambda = 2\pi c/\omega$ and define $\lambda_j = 2\pi c/\omega_j$, n can be written as

$$n(\lambda) = 1 + \frac{Ne^2}{2\varepsilon_0 m} \sum_j \frac{f_j(\lambda^2 - {\lambda_j}^2)\lambda^2 {\lambda_j}^2}{(2\pi c)^2(\lambda^2 - {\lambda_j}^2)^2 + {\gamma_j}^2 \lambda^2 {\lambda_j}^4}$$

At normal dispersion region ${\lambda_j}^4$ term can be omitted in the denominator

$$1 + \frac{Ne^2}{2\varepsilon_0 m} \sum_j \frac{a_j \lambda^2}{\lambda^2 - {\lambda_j}^2}$$





Whole spectrum refractive index
From http://physics.stackexchange.com

Appendix II

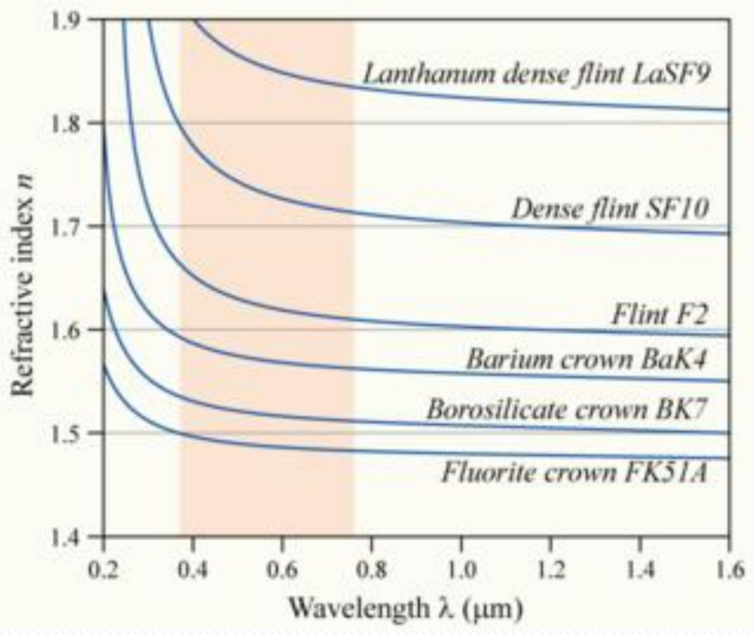
At normal dispersion region $\lambda_1, \lambda_2, \ldots, \lambda_j < \lambda < \lambda_{j+1}, \lambda_{j+2}, \ldots$

$$n = 1 + a_1 + a_2 + \cdots + a_{j-1} + \frac{a_j \lambda^2}{\lambda^2 - {\lambda_j}^2}$$

$$\approx 1 + a_1 + a_2 + \cdots + a_{j-1} + a_j \left[1 + \left(\frac{\lambda_j}{\lambda}\right)^2 + \left(\frac{\lambda_j}{\lambda}\right)^4 + \cdots \right]$$

$$= C + \frac{F}{\lambda^2} + \frac{G}{\lambda^4} + \cdots$$

The last equation is known as the Cauchy equation



The variation of refractive index with wavelength
From Wikipedia

Appendix II