A Physical Pendulum With Damping
Due to Sliding Friction

A physical pendulum is described where the damping of the motion is due to sliding friction in the support. The motion is shown to consist of damped oscillations, with the amplitude decreasing linearly with time until the motion abruptly stops. The objective of the related experimental study is to show that the observed motion is an good agreement with the predicted equations of motion.

I. INTRODUCTION

We consider a pendulum made up of a length, L, of a 2"x4" wooden board suspended by a horizontal circular rod of radius, R, that passes through a square hole cut through the board. The size of the sides of the square hole are barely larger than the diameter of the rod. As the board swings as a pendulum, the points of contact between the square hole and the rod must slide around the rod. This creates a sliding friction that provides the main source of damping of the motion. The configuration is similar to the situation shown in Figure 1.

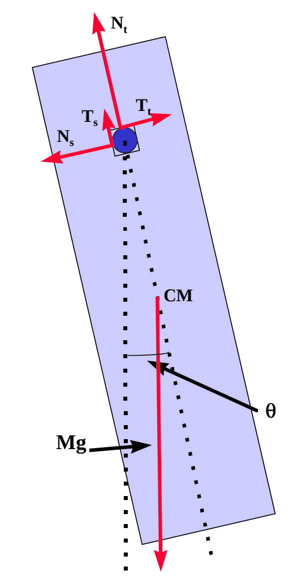

FIG. 1. The forces acting on the physical pendulum.

In this experiment the sonic rangefinder is used to record the oscillatory motion of the pendulum in an ascii file. The measured motion should then be compared in detail with the motion predicted by Newton's laws.

II. APPLICATION OF NEWTON'S LAWS TO THE PENDULUM

We consider the limit where the coefficient of sliding friction between the rod and the square hole is relatively small so that we may assume $\mu_k^2 << 1$. We also assume that the initial angular amplitude of the swing is much less than one radian, that is: $|\theta_0| << 1$, where the angle is in radians. In this limit we may replace $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$. One can write two simultaneous equations relating the components of force parallel and perpendicular to the board to the corresponding components of the acceleration of the center of mass of the rod. For small angles of rotation, the acceleration terms are small and these two equations can be solved for the force on the board due to the supporting rod. From these forces one can derive the torque that will damp the motion of the swinging board as,

$$\tau_f \simeq -\mu_k MgR \quad sgn(d\theta/dt). \tag{1}$$

This torque always tends to slow down the pendulum since it always tend to turn the object in the opposite direction to the direction it is moving.

To describe the oscillation of the pendulum, we use $I\alpha = \sum \tau_i$, where I is the moment of inertia of the rigid body about the axis of rotation, $\alpha = d^2\theta/dt^2$ is the angular acceleration of the rotating rigid body, and the τ_i are the torques due to the external forces acting on the board. One of these forces is due to the force of gravity, which gives a torque equal to $-Mgh\sin\theta$. In the small angle limit this becomes $-Mgh\theta$, where θ is the angle between the vertical and the swinging board, M is the mass of the board, h is the distance between the axis of rotation and the center of mass of the board, and g is the acceleration of a freely falling body under the action of gravity. Using the parallel axis theorem for moments of inertia, we find

$$I = M\left(h^2 + \frac{L^2}{12}\right). \tag{2}$$

Solving for α, we find

$$\frac{d^2\theta}{dt^2} = -\left(\frac{gh}{h^2 + \frac{L^2}{12}}\right)\theta \\ -\mu_k\left(\frac{R}{h}\right)\left(\frac{gh}{h^2 + \frac{L^2}{12}}\right) sgn(d\theta/dt). \tag{3}$$

Newton's laws applied to the rotation of a rigid body have led to a relation between θ and it's derivatives. This relation only holds for maximum angles of swing which are less than 20^o relative to the vertical equilibrium position. To simplify this relation we let,

$$\omega^2 = \frac{gh}{h^2 + L^2/12}, \tag{4}$$

and

$$\Delta\theta_0 = -4\mu_k \left(\frac{R}{h}\right). \tag{5}$$

Equation (3) becomes

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = \left(\frac{\Delta\theta_0}{4}\right)\omega^2 sgn(d\theta/dt). \tag{6}$$

We now use the differential equation for θ to determine how the pendulum moves. Suppose that at $t = 0$ the pendulum starts from rest with $\theta(t = 0) = \theta_0$. As the pendulum swings toward the center $sgn(d\theta/dt) = -1$, so that until the motion stops at some negative value of θ, we have

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = -\left(\frac{\Delta\theta_0}{4}\right)\omega^2. \tag{7}$$

The most general function of time which satisfies the last equation is

$$\theta = A\cos(\omega t + \alpha) - \frac{\Delta\theta_0}{4}, \tag{8}$$

where A and α are constants to be determined by the initial conditions: $\theta(0) = \theta_0$, and $(d\theta/dt)_{t=0} = 0$. Applying the second condition we find that $\alpha = 0$. The first condition gives $\theta_0 = A - (\Delta\theta_0/4)$. We have for the first half of a period of oscillation,

$$\theta = \left(\theta_0 + \frac{\Delta\theta_0}{4}\right)\cos(\omega t) - \frac{\Delta\theta_0}{4}. \tag{9}$$

When $\omega t = \pi$ the object stops at the largest negative value of θ. We examine how much the amplitude of oscillation decreases in a half-period.

$$\theta(\pi/\omega) = -\left(\theta_0 + \frac{\Delta\theta_0}{4}\right) - \frac{\Delta\theta_0}{4} = -\theta_0 - \frac{\Delta\theta_0}{2}. \tag{10}$$

We see that in one-half of a period that the amplitude decreases by $\Delta\theta_0/2$. This decrease does not depend on the amplitude. This is the decrease in amplitude that will occur in any half period, π/ω. Thus, the amplitude decreases at a constant rate with time. This rate is the change in amplitude $\Delta\theta_0/2$ divided by π/ω, the time during which this change occurs.

$$\frac{d\theta_0(t)}{dt} = \frac{\Delta\theta_0\omega}{2\pi} = -2\mu_k(R/h)(\omega/\pi). \tag{11}$$

Thus, the pendulum oscillates back and forth with the same period that would describe the motion without friction. However, in each swing of the pendulum the amplitude decreases by the fixed amount, $y\Delta\theta_0$. This can be put into a single equation,

$$\theta = \left(\theta_0 - \frac{2\mu_k R}{\pi h}\omega t\right)\cos(\omega t + \alpha). \tag{12}$$

III. COMPARING THEORY WITH EXPERIMENT

It is then sensible to fit the motion data to the following functional form

$$y = (a + bx)\sin(cx + d) + e, \quad a + bx > 0, \tag{13}$$

and,

$$y = e, \quad a + bx \leq 0, \tag{14}$$

where y is the distance of some point on the pendulum from the rangefinder, x is the time in the language of the curve fitting program dfita.exe, a is the initial amplitude of the motion, b is a negative number that takes the sliding frictional damping into account, c it the same as ω, d is a phase factor that depends on when the clock was started, and e is the distance from the origin of displacements to the equilibrium point. This is exactly choice # 16 with dfita.exe. We should compare the value that dfita.exe gives for c with what is calculated for ω from Eq. (4). The fit parameters determined by dfita.exe are contained at the end of the file generated by the program. The value of b should be compared with $2\mu_k R\omega L'/(\pi h)$, where L' is the distance from the axis of rotation of the point on the pendulum being observed by the rangefinder. To determine this effective point, measure the angle amplitude when the pendulum is started and determine L' so that $a = L'\theta_0$. To compare b with theory you must make careful measurements of R and the coefficient of kinetic friction between two similar surfaces. Even this is a bit crude, since finding a perfect match between the two surfaces may be crude.

You should record the motion for at least three cases. The differences between the parameters for the three cases will give a good idea about the error in the measurements. A major test of the theory lies in how well the observed data agrees with Eqs. (13) and (14). This is the main prediction of Newton's laws. In addition to the fit parameters dfita.exe writes the root mean square difference between the data and the fit to the output file. Part of this deviation is not a flaw in the theory, but is due to the use of a diffuse reflector with the rangefinder in order not to lose the signal when the angle of reflection would be such that the reflection would miss the detector. This would otherwise occur for the larger amplitude oscillations. The major subtle points and sources of error should be discussed in your laboratory report.

IV. AN EXAMPLE OF DATA ANALYSIS

Finally, we consider a case where the pendulum was pulled back and started from rest with the data being taken by the Sonic Rangefinder. The recorded motion is shown in Figure 2.

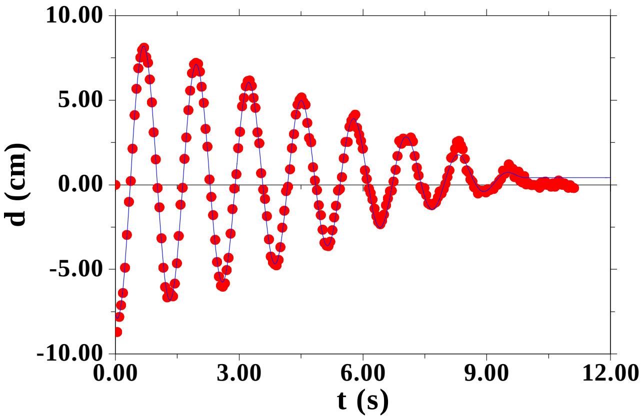

FIG. 2. Measured position as a function of time compared with a curve fit with Equations (13) and (14)

This figure shows the kind of data and the agreement with Eq. (13) that is expected with this study. To compare with the results of the datafit we use $L = 66$ cm, $h = 24.5$ cm, and $R = 0.64$ cm. We estimate the coefficient of friction of smooth steel against smooth steel that is not oiled to be $\mu_k = 0.3$. Calculating from Eq. (3) we get $\omega = 4.99$ /s, and $2\mu_k R\omega/(\pi h) = 0.025$ /s. If we take the distance from the axis of rotation down to the point where the reflection occurs to be 40 cm, we have $L' = 40$ cm, so that our estimate of b in the fit to the data is around $b = -1.0$ cm/s. The values obtained by fitting with dfita.exe are $a = 8.4$ cm, $b = -0.84$ cm/s, $c = 4.93$ /s, $d = -1.83$, and $e = 0.42$ cm. Thus, the value of ω differs from observation by a bit more than 1 % and the functional shape of the decay is amazingly close to the anticipate fitting function. The estimate of the damping rate is good.